Superconvergence of period doubling cascade in trapezoid maps

—— Its Rigorous proof and superconvergence of period doubling cascade starting from a period p solution ——

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Abstract

In the symmetric and the asymmetric trapezoid maps, as a slope a of the trapezoid is increased, the period doubling cascade occurs and the symbolic sequence of periodic points is the Metropolis-Stein-Stein sequence R^{*m} and the convergence of the onset point a_m of the period 2^m solution to the accumulation point a_c is exponentially fast. In the previous paper, we proved these results.

In this paper, we give the detailed description of the proof on the results. Rigorously, we show that

$$
\begin{aligned}
\epsilon_m &= \frac{b a_c^{-2^m} \gamma^{-\zeta_m}}{G'_\infty(a_c)}(1+h_m), \\
\delta_m &= \gamma^{(-1)^m/3}(a_c \gamma^{2/3})^{2^m}(1+l_m), \\
\lim_{m\to\infty} h_m &= 0, \\
\lim_{m\to\infty} l_m &= 0,
\end{aligned}
$$

where $\epsilon_m \equiv a_c - a_m, \delta_m \equiv \frac{\epsilon_m - \epsilon_{m+1}}{\epsilon_{m+1} - \epsilon_{m+2}}$, b and γ are the smaller size of the trapezoid and the ratio of its two slopes, respectively. $\gamma = 1$ corresponds to the symmetric trapezoid. Further, we study the period doubling cascade starting from period $p(\geq 3)$ solution. We show

$$
\begin{aligned}
\epsilon_m &\propto (\gamma^2 a_{p,c}^p)^{-2^m} \gamma^{\zeta_m}, \\
\delta_m &\simeq \gamma^{(-1)^{m-1}/3}(a_{p,c}^p \gamma^{4/3})^{2^m},
\end{aligned}
$$

where $a_{p,c}$ is the accumulation point of the onset of the period $p \times 2^m$ solution.

§1. Introduction

It is well known that in a class of one-dimensional map, as a parameter is changed, the period doubling bifurcation cascades, and there exist several universal properties [1]. One of the universal quantities is the so called Feigenbaum constant δ and it depends only on the exponent z characterizing the behavior of the map around the critical point.

About a decade ago, there had been controversy on the $z \to \infty$ limit of $\delta(z)$. As a typical example, the map $x_{n+1} = f(x_n) = 1 - a|x_n|^z$ has been extensively studied and two different values were conjectured for this limit, one is finite and the other is infinity [2), 3)]. This problem was solved by J. P. Eckmann and H. Epstein and they proved that the limit is finite [4)].

In the previous paper [5)], instead of considering a map with finite z and taking $z \to \infty$, we investigated a map with $z = \infty$. As such maps, we treated the symmetric and asymmetric trapezoid maps.

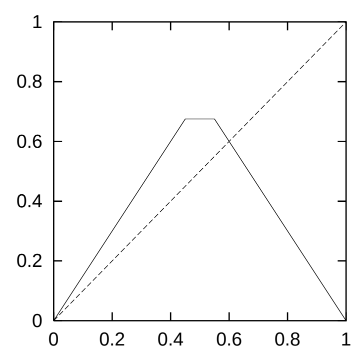

Fig. 1. Trapezoid map $T_{(a,b)}(x)$

The symmetric trapezoid map $x_{n+1} = T_{(a,b)}(x_n)$ defined in $[0, 1]$ is given by

$$T_{(a,b)}(x) = \begin{cases} ax & \text{for } 0 \le x \le (1-b)/2, \\ a(1-b)/2 & \text{for } (1-b)/2 \le x \le (1+b)/2 \ , \\ a(1-x) & \text{for } (1+b)/2 \le x \le 1 \ , \end{cases} \tag{1·1}$$

where $0 < b < 1$ and $a > 0$. See Fig.1.
As a is increased from 1 with fixed b, the successive period doubling bifurcations take place. We assign the symbol L, C and R to an orbit x_i according to the following rule; L for $x_i \in [0, (1-b)/2] \equiv I_L$, C for $x_i \in ((1-b)/2, (1+b)/2) \equiv I_C$, R for $x_i \in [(1+b)/2, 1] \equiv I_R$. We denote this correspondence by $H(x_i)$. Further, we define the conjugate of R or L as $\bar{R} = L$ or $\bar{L} = R$, respectively. Then, in the previous paper [5)] we proved that as a slope a of the trapezoid map is increased, the period doubling bifurcation cascades and the symbolic sequence is the Metropolis-Stein-Stein sequence R^{*m} as usual. And, defining a_m as the onset

of a 2^m-cycle, we calculated $\delta_m \equiv \frac{a_{m+1}-a_m}{a_{m+2}-a_{m+1}}$ and proved that δ_m scales as

$$\delta_m \simeq a_c^{2^m}$$

where a_c is the accumulation point of the period doubling cascade. We called this phenomena the super-convergent period doubling cascade.
Further, we investigated the asymmetric trapezoid map $x_{n+1} = A_{(a,b,\gamma)}(x_n)$ defined in [0,1],

$$A_{(a,b,\gamma)}(x) = \begin{cases} ax & \text{for } 0 \le x \le \alpha, \\ \alpha a & \text{for } \alpha \le x \le \beta\ , \\ \gamma a(1-x) & \text{for } \beta \le x \le 1\ , \end{cases} \tag{1·2}$$

where γ is the ratio of two slopes of the trapezoid, $\alpha = \frac{\gamma}{1+\gamma}(1-b)$ and $\beta = \alpha + b = \frac{b+\gamma}{1+\gamma}$. When $\gamma = 1$, $A_{(a,b,\gamma)}(x)$ is reduced to $T_{(a,b)}(x)$. In the asymmetric map, when γ and b are fixed and a is increased, we proved similar results as in the symmetric map, in particular, as the scaling of δ_m we obtained

$$\delta_m \simeq \gamma^{(-1)^m/3}(a_c\gamma^{2/3})^{2^m}.$$

Finally, we gave approximate expressions for the accumulation point as functions of b which is the length of the smaller side of the trapezoid in both symmetric and asymmetric cases.

In this paper, we give the detailed and complete description of the proofs given in the previous paper. New results in this paper are the scaling relations for the period doubling bifurcation starting with a period $p(\ge 3)$ solution which appears by a tangent bifurcation.

In the following section, we treat the symmetric trapezoid map, and then in the section 3, we treat the asymmetric map. In sections 4 and 5, we study the period doubling bifurcation for period $p(\ge 3)$ solution in symmetric and asymmetric cases, respectively. We give summary and discussion in the last section.

§2. The symmetric case

We consider $T_{\boldsymbol{a}}(x) \equiv T_{(a,b)}(x)$. For brevity we define $\boldsymbol{a} = (a, b)$. Let x_M be the maximum value of $T_{\boldsymbol{a}}(x)$, i.e., $x_M \equiv a\frac{1-b}{2}$. Let $x_{0,1}$ be the non-zero fixed point of $T_{\boldsymbol{a}}(x)$. Then the following lemma is easily proved.

lemma 1

For $1 < a < a_1$, $T_{\boldsymbol{a}}(x)$ has the stable fixed point $x_{0,1} = x_M$, which satisfies $\frac{1-b}{2} < x_M < \frac{1+b}{2}$. a_1 is defined by the equation $x_M = \frac{1+b}{2}$, that is $a_1 = \frac{1+b}{1-b}$.

For $a > a_1$, $T_{\boldsymbol{a}}(x)$ has the unstable fixed point $x_{0,1} = x^* \equiv \frac{a}{a+1}$ and $x^* > \frac{1+b}{2}$. At $a = a_1, x_M = x^*$.

See Fig.2.

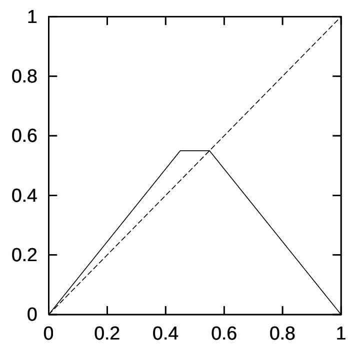

Fig. 2. Trapezoid map $T_{(a,b)}(x)$ for $a = a_1$ with $b = 0.1$

In the region $a > a_1$, by iterating $T_{\boldsymbol{a}}(x)$ twice and rescaling x such that the domain becomes $[0, 1]$, we obtain a trapezoid map with different parameter $\boldsymbol{a}^{(1)}, \boldsymbol{a}^{(1)} = (a^{(1)}, b^{(1)})$.

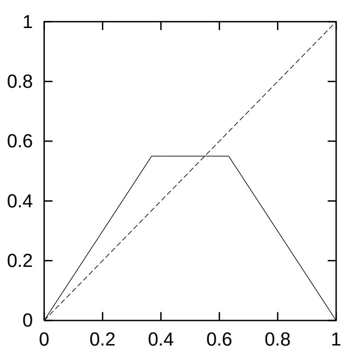

Fig. 3. Trapezoid map $T^2_{(a,b)}(x)$ for $a = a_1$ with $b = 0.1$.

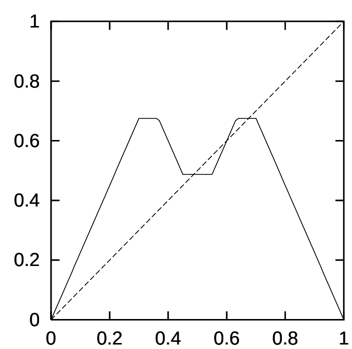

Fig. 4. Trapezoid map $T^2_{(a,b)}(x)$ for $a_1 < a < a_2$ with $b = 0.1$.

See Fig.3 and 4. In fact, we obtain the following lemma.

lemma 2

When $T_{\boldsymbol{a}}(x)$ has the non-zero unstable fixed point $x^* = \frac{a}{a+1} (> \frac{1}{2})$, that is, for $a > a_1$, by rescaling the coordinate x as $x^{(1)} = \frac{x^*-x}{2x^*-1}$ and restricting x to the interval $[1-x^*, x^*]$, $T^2_{\boldsymbol{a}}(x)$ is transformed to $T_{\boldsymbol{a}^{(1)}}(x^{(1)})$ which is defined for $x^{(1)}$ in $[0, 1]$. Here $\boldsymbol{a}^{(1)} \equiv \boldsymbol{\varphi}(\boldsymbol{a}) \equiv (a^2, u(a)b)$ and $u(a) = \frac{a+1}{a-1}$.

The proof is straightforward. $\boldsymbol{\varphi}(\boldsymbol{a})$ is defined as long as $a \neq 1$. In this paper, we restrict ourselves to the case of $a > 1$.

Now, we define the m-th iteration of $\boldsymbol{\varphi}$. That is,

$$\boldsymbol{a}^{(m)} \equiv (a^{(m)}, b^{(m)}) \equiv \boldsymbol{\varphi}^m(\boldsymbol{a}) = (a^{2^m}, u_m(a)b), \qquad (2\cdot1)$$
$$u_m(a) \equiv \prod_{l=0}^{m-1} u(a^{2^l}) = \prod_{l=0}^{m-1} \frac{a^{2^l}+1}{a^{2^l}-1}.$$

These are defined as long as $a \neq 1$. For $a > 1$, as functions of a, $a^{(m)}$ is a continuous strictly increasing function and $b^{(m)}$ is a continuous strictly decreasing function. $\lim_{a\to\infty} b^{(m)} = b < 1$ for any $m > 0$. Let us prove the following lemma.

lemma 3

For any positive integer m, there exists the unique value of $a = a_m$ such that

$$b^{(m)}(a_m) = 1. \qquad (2\cdot2)$$

$\{a_m\}_{m=1}^{\infty}$ is an increasing sequence, i.e., $1 < a_1 < a_2 < \cdots$. Further, the relation $b < b^{(m)} < 1$ holds for $a_m < a$ for $m \geq 1$.

Proof

Let us consider the case of $m = 1$.
$b^{(1)}(a) = 1$ has the unique solution $a = \frac{1+b}{1-b} > 1$, and this is a_1 defined in lemma 1.

Next, let us assume that $b^{(m)}(a_m) = 1$ and $a_m > 1$ for $m \geq 1$. Since $b^{(m+1)}(a) = \frac{a^{(m)}+1}{a^{(m)}-1} b^{(m)}(a)$, $b^{(m+1)}(a_m) > 1$ follows. Thus, there exists the unique value of $a = a_{m+1} (> a_m)$ such that $b^{(m+1)}(a_{m+1}) = 1$.

Therefore, from the mathematical induction, $b^{(m)}(a) = 1$ has a unique solution a_m for any positive integer m and $\{a_m\}_{m=1}^{\infty}$ is an increasing sequence. The inequality $b < b^{(m)} < 1$ for $a_m < a$ is immediately follows from the fact that the function $b^{(m)}(a)$ is strictly decreasing. Q.E.D.

For positive integer m, we define $T_{\boldsymbol{\varphi}^m(\boldsymbol{a})}(x^{(m)})$ for $a > a_m$ from $T_{\boldsymbol{\varphi}^{m-1}(\boldsymbol{a})}(x^{(m-1)})$ successively by the same procedure as in the lemma2.
We define $x_M^{(m)} \equiv a^{(m)} \frac{1-b^{(m)}}{2}$, which is the maximum value of $T_{\boldsymbol{\varphi}^m(\boldsymbol{a})}(x^{(m)})$. Further, we define $x^{(m)*} \equiv \frac{a^{(m)}}{a^{(m)}+1}$. $x_M^{(0)} = x_M$ and $x^{(0)*} = x^*$.

lemma 4

For any non-negative integer m and for $a_m < a$ there exists a unique non-zero fixed point for $T_{\boldsymbol{\varphi}^m(\boldsymbol{a})}(x^{(m)})$ which is defined in $[0, 1]$. For $a_m < a < a_{m+1}$, $\frac{1-b^{(m)}}{2} < x_M^{(m)} < \frac{1+b^{(m)}}{2}$ and

$x_M^{(m)}$ is the stable fixed point of $T_{\boldsymbol{\varphi}^m(\boldsymbol{a})}(x^{(m)})$. At $a = a_{m+1}$, $x_M^{(m)} = x^{(m)*}$. For $a_{m+1} < a$, the non-zero fixed point for $T_{\boldsymbol{\varphi}^m(\boldsymbol{a})}(x^{(m)})$ is $x^{(m)*}$ and unstable, and $x^{(m)*} > \frac{1+b^{(m)}}{2}$. Here, we define $x^{(0)} \equiv x, a_0 \equiv 1, a^{(0)} \equiv a$ and $b^{(0)} \equiv b$.

Proof

First of all, we notice that for $m \geq 1$ $T_{\boldsymbol{\varphi}^m(\boldsymbol{a})}(x^{(m)})$ is really a trapezoid map because from lemma 3 $b < b^{(m)} < 1$ for $a_m < a$ and for $m \geq 1$.

For the case $m = 0$, the statement follows from lemma 1.

Let us assume that the statement holds for $m \geq 0$. Since $x^{(m)*} = \frac{a^{(m)}}{a^{(m)}+1}$ is the unstable fixed point of $T_{\boldsymbol{a}^{(m)}}(x^{(m)})$ for $a_{m+1} < a$, from lemma 2, taking $T^2_{\boldsymbol{a}^{(m)}}(x^{(m)})$ and rescaling $x^{(m)}$ as $x^{(m+1)} = \frac{x^{(m)*}-x^{(m)}}{2x^{(m)*}-1}$, we obtain $T_{\boldsymbol{a}^{(m+1)}}(x^{(m+1)})$ which is defined in $[0,1]$. The following equivalence relations are easily proved for $m \geq 0$,

$$b^{(m+1)}(a) = 1 \Longleftrightarrow x_M^{(m)} = \frac{1+b^{(m)}}{2} \Longleftrightarrow a^{(m)} = \frac{1+b^{(m)}}{1-b^{(m)}}. \tag{2·3}$$

Let us consider $x_M^{(m+1)} = a^{(m+1)}\frac{1-b^{(m+1)}}{2}$. From the relation (5), since $b^{(m+2)}(a_{m+2}) = 1$ it follows that $x_M^{(m+1)} = \frac{1+b^{(m+1)}}{2}$ at $a = a_{m+2}$. Since $x_M^{(m+1)}$ is the strictly increasing function w.r.t. a, we obtain $0 < \frac{1-b^{(m+1)}}{2} < x_M^{(m+1)} < \frac{1+b^{(m+1)}}{2}$ for $a_{m+1} < a < a_{m+2}$. This implies $x_M^{(m+1)}$ is the unique non-zero fixed point for $T_{\boldsymbol{\varphi}^{m+1}(\boldsymbol{a})}(x^{(m+1)})$ and stable. It is easily shown that for $a_{m+2} < a$, $x_M^{(m+1)}$ is no more fixed point but $x^{(m+1)*} = \frac{a^{(m+1)}}{a^{(m+1)}+1}$ becomes unstable fixed point and $x^{(m+1)*} > \frac{1+b^{(m+1)}}{2}$. At $a = a_{m+2}, a_M^{(m+1)} = x^{(m+1)*}$ holds. Thus, by the mathematical induction, the proof completes.

lemma 5

The unique non-zero fixed point of $T_{\boldsymbol{\varphi}^m(\boldsymbol{a})}(x^{(m)})$ in lemma 4 is a member of a periodic cycle with prime period 2^m of $T_{\boldsymbol{a}}(x)$.

Proof

The case of $m = 0$, it is trivial. Now, let us fix $m(\geq 0)$ and assume that for $0 \leq k \leq m$ the unique non-zero fixed point of $T_{\boldsymbol{\varphi}^k(\boldsymbol{a})}(x^{(k)})$ which exists for $a_k < a$ is the member of the cycle with prime period 2^k. Thus, from lemma 4 for $a_{m+1} < a$ the unique non-zero fixed point of $T_{\boldsymbol{\varphi}^{m+1}(\boldsymbol{a})}(x^{(m+1)})$ does not correspond to any periodic points with period less than 2^{m+1} because it is stable for $a_{m+1} < a < a_{m+2}$. Thus, the non-zero fixed point of $T_{\boldsymbol{\varphi}^{m+1}(\boldsymbol{a})}(x^{(m+1)})$ is the member of the cycle with prime period 2^{m+1}. This completes the proof.

Thus, we obtain the following theorem.

Theorem 1

$x = 0$ is period 1 solution of $T_{\boldsymbol{a}}(x)$ for any $a > 0$. Except for this, for any non-negative integer m, for $a_m < a < a_{m+1}$ there is the unique cycle with prime period 2^k for $k = 0, 1, \cdots, m$ and no other periodic points exist. The periodic solution with prime period 2^m is stable for $a_m < a < a_{m+1}$ and unstable for $a_{m+1} < a$. At $a = a_{m+1}$, the period 2^m cycle and the period 2^{m+1} cycle coincide.

Proof

For the case of $m = 0$, the statement is trivial. Let us assume that the statement holds for $0, 1, \cdots, m$. Then we only have to prove that for $a_{m+1} < a < a_{m+2}$, except for the period 2^k cycles for $k = 1 \sim m+1$, there is no other cycle. Let us consider any initial point x in (0, 1) which does not corresponds to the unstable cycles of period $2^k, (k = 0 \sim m)$. Then, the orbit $T^i_{\boldsymbol{a}}(x)$ finally enters the domain of the map $T_{\boldsymbol{\varphi}^m(\boldsymbol{a})}(x^{(m)})$ if i is appropriately chosen. Thus, since any point except for $x^{(m)} = 0$ and 1 converges to the fixed point of $T_{\boldsymbol{\varphi}^m(\boldsymbol{a})}(x^{(m)})$ for $a_{m+1} < a < a_{m+2}$, the nonexistence of other periodic points for $a_{m+1} < a < a_{m+2}$ follows. The latter part of the theorem follows from lemma 4 immediately.

From the above arguments we conclude that as a is increased from 1, the period doubling bifurcation cascades, and at $a = a_m$ the periodic solution with prime period 2^m appears for $m \geq 0$.

Now, let us investigate the upper bound of the sequence $\{a_m\}$. $x_M^{(m)}(a) = a^{(m)}\frac{1-b^{(m)}}{2}$ is defined for $a > 1$ and the strictly increasing function w.r.t a. For any $m > 0, x_M^{(m)}(a_m) = 0$ and $\lim_{a\to\infty} x_M^{(m)}(a) = \infty$ follows. Thus, for $m > 0$ we define $a_M(m)$ as the unique solution of the equation

$$x_M^{(m)}(a) = 1. \qquad (2\cdot4)$$

For $m = 0$, we define $a_M(0) = \frac{2}{1-b}$. We abbreviate this as a_M. Then, we obtain the following lemma.

lemma 6

$$\begin{aligned} &1 < a_1 < a_2 < \cdots < a_m < a_{m+1} < \cdots \\ &< a_M(m+1) < a_M(m) \cdots < a_M(2) < a_M(1) < a_M. \end{aligned} \qquad (2\cdot5)$$

Proof

For $m > 0$, since $x_M^{(m)}(a_m) = 0, a_m < a_M(m)$ follows. For $m \geq 0$, $x_M^{(m+1)}(a)$ is rewritten as

$$x_M^{(m+1)}(a) = \{a^{(m)}\}^2 \frac{1}{a^{(m)} - 1}(x_M^{(m)}(a) - \frac{1 + b^{(m)}}{2}). \qquad (2\cdot6)$$

Then, $x_M^{(m+1)}(a_M(m)) = \frac{a^{(m)}}{a^{(m)}-1} > 1$. Therefore, $a_M(m+1) < a_M(m)$ for $m \geq 0$. Thus, we obtain $a_m < a_{m+1} < a_M(m+1) < a_M(m)$ for $m \geq 0$. Q.E.D.

From this,

$$a_c \equiv \lim_{m\to\infty} a_m \leq \lim_{m\to\infty} a_M(m) < a_M \tag{2·7}$$

follows. Therefore a_c is finite.

Now, let us investigate the symbolic sequence. The sequence of R and L for the period 2^m solution becomes Metropolis-Stein-Stein(MSS) sequence. Let us prove this.

For $m \geq 0$, we define the onset of the 2^n-cycle for $T_{\boldsymbol{\varphi}^m(\boldsymbol{a})}(x^{(m)})$ as $a_n^{(m)}$ and the i-th orbit of the period 2^n cycle as $x_{n,i}^{(m)}$. $a_n^{(0)} = a_n$. For $m = 0$, we often omit the superscript (0). As $x_{m,1}$ we set the largest x value among 2^m members of the periodic cycle. That is, $x_{m,1} = x_M = a(1-b)/2$ when the cycle is stable. Further, for $m \geq 0$ we divide the coordinate space $[0,1]$ of $x^{(m)}$ into the three intervals as $I_L^{(m)} \equiv [0, \frac{1-b^{(m)}}{2}], I_C^{(m)} \equiv (\frac{1-b^{(m)}}{2}, \frac{1+b^{(m)}}{2})$ and $I_R^{(m)} \equiv [\frac{1+b^{(m)}}{2}, 1]$.

lemma 7

For any positive integer m, when $a_m < a$,

1. $I_C^{(m)} \ni x^{(m)}$ is equivalent to $I_C^{(m-1)} \ni x^{(m-1)}$,
2. if $I_L^{(m)} \ni x^{(m)}$ then $I_R^{(m-1)} \ni x^{(m-1)}$,
3. if $I_R^{(m)} \ni x^{(m)}$ then $I_L^{(m-1)} \ni x^{(m-1)}$,

where $x^{(m)}$ and $x^{(m-1)}$ are related by the coordination transformation used to define $T_{\boldsymbol{\varphi}^m(\boldsymbol{a})}(x^{(m)})$ from $T_{\boldsymbol{\varphi}^{m-1}(\boldsymbol{a})}(x^{(m-1)})$.

proof

For $a_m < a$, $x^{(m)} = 0, \frac{1-b^{(m)}}{2}, \frac{1+b^{(m)}}{2}$ and 1 correspond to $x^{(m-1)} = x^{(m-1)*}, \frac{1+b^{(m-1)}}{2}, \frac{1-b^{(m-1)}}{2}$ and $1 - x^{(m-1)*}$, respectively. Since this correspondence is linear, the statements hold.

lemma 8

The fixed point $x_{0,1}^{(m)}$ of $T_{\boldsymbol{\varphi}^m(\boldsymbol{a})}(x^{(m)})$ corresponds to $x_{m,2^m}$.

proof

Let $x_{m,i}$ be the orbit of 2^m solution corresponding to $x_{0,1}^{(m)}$. From lemma 4, for $a_m < a < a_{m+1}, x_{0,1}^{(m)} = x_M^{(m)}$ and $I_C^{(m)} \ni x_{0,1}^{(m)}$. Then, from lemma 7, $I_C \ni x_{m,i}$ follows. Therefore, $x_{m,i}$ is mapped to x_M by $T_{\boldsymbol{a}}$, which is $x_{m,1}$. Thus, $i = 2^m$.

lemma 9

For any positive integer m, symbols $H(x_{m,i})$ for the number of 2^m cycle satisfy the followings.

1. $H(x_{m,i})$ does not change for $a_m \le a$ and for $i = 1, \cdots, 2^m - 1$.
2. For even m,
$$H(x_{m,2^m}) = \begin{cases} L & \text{for } a = a_m, \\ C & \text{for } a_m < a < a_{m+1} \ , \\ R & \text{for } a_{m+1} \le a \ . \end{cases}$$
and for odd m,
$$H(x_{m,2^m}) = \begin{cases} R & \text{for } a = a_m, \\ C & \text{for } a_m < a < a_{m+1} \ , \\ L & \text{for } a_{m+1} \le a \ . \end{cases}$$
3. $\{H(x_{m,i})\}$ is the MSS sequence R^{*m} * for $a_{m+1} \le a$.

proof

Let us consider the case of $m = 1$. At $a = a_1$, $x_{0,1} = \frac{1+b}{2} \in I_R$. Then, at $a = a_1, H(x_{0,1}) = H(x_{1,1}) = H(x_{1,2}) = R$. For $a_1 < a < a_2$, $x_{0,1}^{(1)} = x_M^{(1)} = \frac{a^{(1)}(1-b^{(1)})}{2}$ and for $a_2 \le a, x_{0,1}^{(1)} = x^{(1)*}$. From lemma 8, $x_{0,1}^{(1)}$ corresponds to $x_{1,2}$. Thus, from lemma 7, $H(x_{1,2})$ is C for $a_1 < a < a_2$, and is L for $a_2 \le a$. On the other hand, for $a_1 \le a$, since $x_{1,1} \ge x_{0,1} = \frac{a}{a+1}$ then $H(x_{1,1}) = R$. Therefore, $H(x_{1,1})H(x_{1,2}) = RL$ for $a_2 \le a$. Therefore, for $m = 1$, the lemma holds.

Next, we assume that the lemma holds for the case of $m(\ge 1)$. At $a = a_{m+1}$, $x_{m+1,i}$ and $x_{m+1,i+2^m}$ emerge from $x_{m,i}(i = 1 \sim 2^m)$. Then, at $a = a_{m+1}$, $H(x_{m,i}) = H(x_{m+1,i}) = H(x_{m+1,i+2^m})$ for $i = 1, \cdots, 2^m$. These are L or R. Let us assume that at some value of $a(> a_{m+1})$, $H(x_{m+1,i})$ becomes C. Then, $x_{m+1,i}$ is mapped to x_M by $T_{\boldsymbol{a}}$. Thus, in this case, $a < a_{m+2}$ and $x_{m+1,i+1} = x_M = x_{m+1,1}$ follow. So, i should be 2^{m+1}. Thus, $x_{m+1,i}(i = 1 \sim 2^{m+1} - 1)$ does not change its symbol for $a_{m+1} \le a$. From lemma 8 $x_{m+1,2^{m+1}}$ corresponds to $x_{0,1}^{(m+1)} = x_M^{(m+1)}$ for $a_{m+1} \le a \le a_{m+2}$. Thus, $H(x_{m+1,2^{m+1}}) = C$ for $a_{m+1} < a < a_{m+2}$. For $a_{m+2} \le a$, $x_{0,1}^{(m+1)} = x^{(m+1)*} \in I_R^{(m+1)}$. Thus, from lemma 7, for $a_{m+2} \le a$,

$$H(x_{m+1,2^{m+1}}) = \begin{cases} L & \text{for odd } m+1, \\ R & \text{for even } m+1. \end{cases}$$

Therefore, the statements 1 and 2 hold for $m + 1$.
At $a = a_{m+1}, H(x_{m+1,2^{m+1}}) = H(x_{m,2^m})$. Then, from the assumption,

$$H(x_{m+1,2^{m+1}}) = \begin{cases} R & \text{for odd } m+1, \\ L & \text{for even } m+1. \end{cases}$$

* Strictly speaking, the last symbol of the sequence in our definition is R or L and is different from that in the MSS sequence, C

Let $H(x_{m,1})H(x_{m,2})\cdots H(x_{m,2^m})$ be the MSS sequence for $a \geq a_{m+1}$. From the above argument, for $a \geq a_{m+2}$, the sequence for 2^{m+1} cycle is

$$
\begin{aligned}
&H(x_{m,1})H(x_{m,2})\cdots H(x_{m,2^m-1})H(x_{m,2^m}) \\
&\times H(x_{m,1})H(x_{m,2})\cdots H(x_{m,2^m-1})\overline{H(x_{m,2^m})}.
\end{aligned}
$$

This is the MSS sequence. Thus, the statement 3 is proved. This completes the proof.

Now, we derive the equations for which a_m and a_c should satisfy, respectively, and obtain the asymptotic expression for δ_m. First, we assign 0 or 1 to any orbit x_i with the symbol L or R, respectively. We denote this correspondence as $s_i = s(x_i)$. Further, we define the function $f_s(x)$ for $x \in I_L \cup I_R$ as

$$
f_s(x) = sa + (1-2s)ax, \tag{2·8}
$$

where $s = s(x)$. Starting from the maximum value of $T_{a,b}(x)$, $x_1 = x_M = a(1-b)/2$, if $x_1, x_2, \cdots, x_{n-1} \in I_L \cup I_R$, x_n is expressed as

$$
\begin{aligned}
x_n &= f_{s_{n-1}}(x_{n-1}) = f_{s_{n-1}} \circ f_{s_{n-2}} \circ \cdots \circ f_{s_1}(x_1) = \sum_{l=1}^{n} \xi_l a^l, \qquad (2·9)\\
\xi_l &= s_{n-l} \prod_{j=n-l+1}^{n-1} (1-2s_j) \ \text{ for } \ 2 \leq l \leq n-1, \\
\xi_1 &= s_{n-1}, \quad \xi_n = \frac{1-b}{2} \prod_{j=1}^{n-1} (1-2s_j).
\end{aligned}
$$

For $m \geq 1$ let us consider the period 2^m solution $\{x_{m,i}\}$ for $a_m \leq a \leq a_{m+1}$.

$$
\begin{aligned}
x_{m,2^m} &= f_{s_{2^m-1}}(x_{m,2^m-1}) \equiv \sum_{l=0}^{2^m-1} c_l^{(m)} a^{2^m-l} \equiv F_m(a), \qquad (2·10)\\
c_l^{(m)} &= s_l \prod_{j=l+1}^{2^m-1} (1-2s_j) = \xi_{2^m-l}, \ \text{ for } \ 1 \leq l \leq 2^m-2, \\
c_0^{(m)} &= \frac{1-b}{2} \prod_{j=1}^{2^m-1} (1-2s_j) = s_0 \prod_{j=1}^{2^m-1} (1-2s_j), \\
c_{2^m-1}^{(m)} &= s_{2^m-1},
\end{aligned}
$$

where $s_0 \equiv \frac{1-b}{2}$. Note that $F_m(a)$ is determined by the sequence $(s_1, \cdots, s_{2^m-1})$ and is independent of s_{2^m}. From lemma 9, the symbols $H(x_{m,i})(i = 1 \sim 2^m-1)$ do not change for $a_m \leq a$. Further, for any positive integer j these symbols are equal to the first $2^m - 1$

symbols of the period 2^{m+j} solution when $a_{m+j} \leq a$. Therefore, $F_m(a)$ expresses $x_{2^{m+j},2^m}$ in the region $[a_{m+j}, a_{m+j+1}]$. From the statement 2 in lemma 9, the following relations follow

$$x_{m,2^m}(a_m) = \frac{1+(-1)^{m-1}b}{2}, \tag{2·11}$$

$$x_{m,2^m}(a_{m+1}) = \frac{1+(-1)^{m}b}{2}. \tag{2·12}$$

Thus, we obtain the following conditions for a_m,

$$F_m(a_m) = \frac{1+(-1)^{m-1}b}{2}, \tag{2·13}$$

or

$$F_{m-1}(a_m) = \frac{1+(-1)^{m-1}b}{2}. \tag{2·14}$$

Since the symbolic sequence is the Metropolis-Stein-Stein sequence R^{*m},

$$\Pi_{j=1}^{2^m-1}(1-2s_j) = (-1)^m, \tag{2·15}$$

$$s_{2^m} = \frac{1+(-1)^m}{2} \tag{2·16}$$

follow. From these relations, for $l \geq 0$ we obtain

$$r_l \equiv \{\prod_{j=1}^{2^m-1}(1-2s_j)\}^{-1}c_l^{(m)} = (-1)^m c_l^{(m)} = s_l \prod_{j=1}^{l}(1-2s_j).$$

That is, r_l is m-independent as long as it is defined. Thus, we get

$$r_l = s_l \prod_{j=1}^{l}(1-2s_j) \text{ for any } l(\geq 0), \tag{2·17}$$

$$r_0 = (1-b)/2, \quad r_{2^m} = -(1+(-1)^m)/2 \text{ for any } m(\geq 0), \tag{2·18}$$

and for $l > 0$ the successive values (r_l, r_{l+1}) take the following six sets of values, $(0, \pm 1), (\pm 1, 0), (1, -1), (-1, 1)$. * Let us define

$$G_m(a) \equiv (-1)^m a^{-2^m} F_m(a) = \sum_{l=0}^{2^m-1} r_l a^{-l}. \tag{2·19}$$

$$G_\infty(z) \equiv \lim_{m\to\infty} G_m(z). \tag{2·20}$$

Then, $G_\infty(z)$ is the analytic function for $|z| > 1$. The equation (2·13) becomes

$$G_m(a_m) = a_m^{-2^m}\frac{(-1)^m - b}{2}, \tag{2·21}$$

* In paper [5], two cases $(0, -1)$ and $(-1, 0)$ are missing.

and the accumulation point a_c satisfies the equation,

$$G_\infty(a_c) = 0. \tag{2·22}$$

Let us estimate a_m for large m. Putting $a_m = a_c - \epsilon_m$ and using the mean value theorem, the right hand side of eq.(2·21) is rewritten as

$$a_m^{-2^m} \frac{(-1)^m - b}{2} = (a_c^{-2^m} + 2^m \hat{a}_m^{-2^m-1} \epsilon_m) \frac{(-1)^m - b}{2},$$

where $a_m < \hat{a}_m < a_c$. On the other hand, the left hand side of eq.(2·21) is expressed as

$$G_m(a_m) = G_m(a_c) - G'_m(\bar{a}_m)\epsilon_m,$$

where $a_m < \bar{a}_m < a_c$. Thus, we obtain from eq.(2·22)

$$\epsilon_m = \{G_m(a_c) + \frac{b - (-1)^m}{2} a_c^{-2^m}\}(1 + \bar{h}_m)/G'_\infty(a_c) \tag{2·23}$$

where $\bar{h}_m = G'_\infty(a_c)/(G'_m(\bar{a}_m) + 2^m(\hat{a}_m)^{-2^m-1}\frac{(-1)^m-b}{2}) - 1$ and $\lim_{m\to\infty} \bar{h}_m = 0$. Here, we assume $G'_\infty(a_c) \neq 0$, which is proved later. Equation $(2 \cdot 22)$ is rewritten as

$$G_\infty(a_c) = G_m(a_c) + \sum_{l=0}^{\infty} r_{2^m+l} a_c^{-l-2^m} = 0. \tag{2·24}$$

Using the relation

$$r_{2^m+l} = -r_l \ \text{ for } 1 \le l \le 2^m - 1, \tag{2·25}$$

eq.$(2 \cdot 24)$ is further changed to the following.

$$G_\infty(a_c) = G_m(a_c)(1 - a_c^{-2^m}) + a_c^{-2^m}(r_0 + r_{2^m} + a_c^{-2^m} \sum_{l=0}^{\infty} r_{2^{m+1}+l} a_c^{-l}) = 0. \tag{2·26}$$

Thus, we get

$$G_m(a_c) = \frac{b + (-1)^m}{2} a_c^{-2^m} + (a_c^{-2^m})^2 q_m, \tag{2·27}$$

where $q_m = \frac{1}{1-a_c^{-2^m}}(\frac{b+(-1)^m}{2} - \sum_{l=0}^{\infty} r_{2^{m+1}+l} a_c^{-l})$ and $|q_m| < \frac{a_c(2a_c-1)}{(a_c-1)^2}$. Substituting eq.(2·27) into eq.(2·23) we obtain,

$$\epsilon_m = \frac{b a_c^{-2^m}}{G'_\infty(a_c)}(1 + h_m), \tag{2·28}$$

where $h_m = \bar{h}_m(1 + \frac{q_m}{b} a_c^{-2^m}) + \frac{q_m}{b} a_c^{-2^m}$ and $\lim_{m\to\infty} h_m = 0$. Thus,

$$\begin{aligned} \delta_m = \frac{\epsilon_m - \epsilon_{m+1}}{\epsilon_{m+1} - \epsilon_{m+2}} &= a_c^{2^m}(1 + l_m), \\ l_m &= [h_m - h_{m+1} - a_c^{-2^m}\{1 + h_{m+1} - a_c^{-2^m}(1 + h_{m+2})\}] \\ &\quad /[1 + h_{m+1} - a_c^{-2^{m+1}}(1 + h_{m+2})], \end{aligned} \tag{2·29}$$

and $\lim_{m\to\infty} l_m = 0$.

Next, we give the alternative relation for the onset point a_m. As is shown in lemma 3, at $a = a_m$ we have the following equation (2·2) for $m > 0$,

$$b^{(m)}(a_m) = u_m(a_m)b = 1.$$

Defining $v(a) \equiv \frac{a-1}{a+1} = 1/u(a)$ and $v_m(a) \equiv 1/u_m(a) = \prod_{l=0}^{m-1} v(a^{2^l})$, we obtain for $m > 0$

$$v_m(a_m) = b. \tag{2·30}$$

For $m \geq 1$, $v_m(a)$ is rewritten as

$$v_m(a) = \frac{1-a^{-1}}{1+a^{-2^{m-1}}} \prod_{l=0}^{m-2}(1-a^{-2^l}), \tag{2·31}$$

$$\prod_{l=0}^{-1}(1-a^{-2^l}) \equiv 1, \quad v_1(a) = v(a).$$

As is easily shown, for $m \geq 2$, $G_{m-1}(a)$ is expressed by $v_m(a)$ as follows,

$$G_{m-1}(a) = \frac{1}{2}\{(1+a^{-2^{m-1}})v_m(a) - b - (-1)^m a^{-2^{m-1}}\}. \tag{2·32}$$

See Appendix A. Then the equation obtained from the equation $(2 \cdot 14)$

$$G_{m-1}(a_m) = (-1)^{m-1} a_m^{-2^{m-1}} \frac{1+(-1)^{m-1}b}{2} \tag{2·33}$$

is equivalent to the eq.(2·30). From the relation (2·32),

$$G_\infty(a) = \frac{1}{2}(v_\infty(a) - b) \tag{2·34}$$

follows for $a > 1$. Then $G_\infty(a_c) = 0$ is equivalent to $v_\infty(a_c) = b$ which follows from eq.(2·30) immediately. Since

$$v'_\infty(a) = 2v_\infty(a) \sum_{l=0}^{\infty} \frac{2^l a^{2^l-1}}{a^{2^{l+1}}-1}, \tag{2·35}$$

we obtain

$$v'_\infty(a) > 0, \;\; G'_\infty(a) > 0 \;\text{ for } a > 1. \tag{2·36}$$

Putting $a = a_c$ in the eq.(2·35), we get

$$v'_\infty(a_c) = 2b\tau(a_c), \tag{2·37}$$

$$\tau(a_c) \equiv \sum_{l=0}^{\infty} \frac{2^l a_c^{2^l-1}}{a_c^{2^{l+1}}-1} > 0. \tag{2·38}$$

Thus,

$$G'_\infty(a_c) = v'_\infty(a_c)/2 = b\tau(a_c) > 0. \tag{2·39}$$

Therefore, substituting this expression(2·38) into eq.(2·28) we obtain

$$\epsilon_m = \frac{a_c^{-2^m}}{\tau(a_c)}(1+h_m). \tag{2·40}$$

§3. The asymmetric case

As in the symmetric case, we can discuss the period doubling cascade when a is increased with fixed b and γ in the asymmetric case. For brevity we define $\boldsymbol{a} = (a, b, \gamma)$. α and β are expressed as

$$\alpha = \frac{\gamma(1-b)}{1+\gamma}, \quad \beta = \alpha + b = \frac{b+\gamma}{1+\gamma}.$$

α, β and γ are related by $\alpha = \gamma(1-\beta)$. As long as $0 < b < 1$, it holds that $0 < \alpha < 1$ and $0 < \beta < 1$. Let us define $a_L \equiv a$ and $a_R \equiv \gamma a$. Let x_M be the maximum value of $A_{\boldsymbol{a}}(x)$, i.e., $x_M \equiv a\alpha$. We define $x^* \equiv \frac{\gamma a}{1+\gamma a}$. And let $x_{0,1}$ be the non-zero fixed point of $A_{\boldsymbol{a}}(x)$. The following lemma is easily proved.

lemma 1'

1. The case of $\beta \geq 1/2$. For $1 < a < a_1$, $A_{\boldsymbol{a}}(x)$ has the stable fixed point $x_{0,1} = x_M$, which satisfies $\alpha < x_M < \beta$. a_1 is defined by the equation $x_1 = \beta$, that is $a_1 = \frac{\beta}{\alpha}$.
2. The case of $\beta < 1/2$. For $1 < a < \frac{\beta}{\alpha}$, $A_{\boldsymbol{a}}(x)$ has the stable fixed point $x_{0,1} = x_M$, which satisfies $\alpha < x_M < \beta$. For $a \geq \frac{\beta}{\alpha}$ the non-zero fixed point becomes x^* and continues to be stable until $\frac{1-\beta}{\alpha}$. Thus, in this case we define $a_1 = \frac{1-\beta}{\alpha} = \frac{1}{\gamma}$.

In both cases, for $a > a_1 (> 1)$, $A_{\boldsymbol{a}}(x)$ has the unstable fixed point $x_{0,1} = x^* > \beta$.

In the region $a > a_1$, by iterating $A_{\boldsymbol{a}}(x)$ twice and rescaling x, we obtain a trapezoid map with different parameter $\boldsymbol{a}^{(1)}, \boldsymbol{a}^{(1)} = (a^{(1)}, b^{(1)}, \gamma^{(1)})$. We obtain the following lemma.

lemma 2'

When $A_{\boldsymbol{a}}(x)$ has the non-zero unstable fixed point of $x^* = \frac{\gamma a}{\gamma a+1} (> \beta)$, by rescaling the coordinate x as $x^{(1)} = \frac{x^*-x}{\Delta x}$, $A^2_{\boldsymbol{a}}(x)$ is transformed to $A_{\boldsymbol{a}^{(1)}}(x^{(1)})$ which is defined for $x^{(1)}$ in $[0, 1]$, where several parameters and variables are defined as

$$\begin{aligned}
&\Delta x = \gamma(a-1)/(1+\gamma a), \quad \boldsymbol{a}^{(1)} = \boldsymbol{\varphi}(\boldsymbol{a}), \\
&\boldsymbol{\varphi} \;:\; (a, b, \gamma) \to (a^{(1)}, b^{(1)}, \gamma^{(1)}), \\
&a^{(1)} = a_R^2 = (\gamma a)^2, \quad a_R^{(1)} = a_L a_R = \gamma a^2, \\
&b^{(1)} = u(a,\gamma) b, \quad u(a,\gamma) \equiv \frac{\gamma a + 1}{\gamma(a-1)}, \quad \gamma^{(1)} = 1/\gamma, \quad \alpha^{(1)} = \frac{\alpha a - \beta}{\gamma(a-1)}.
\end{aligned} \tag{3·1}$$

The proof is straightforward. Regardless of the value of β, $\beta^{(1)} > 1/2$ follows. Now, we define the m-th iteration of $\boldsymbol{\varphi}$ for $m \geq 0$. That is,

$$\begin{aligned}
&\boldsymbol{a}^{(m)} \equiv \boldsymbol{\varphi}^m(\boldsymbol{a}) \equiv (a^{(m)}, b^{(m)}, \gamma^{(m)}), \\
&a^{(m)} = \gamma^{2(2^m-(-1)^m)/3} a^{2^m}, \quad a_R^{(m)} = \gamma^{(2^{m+1}+(-1)^m)/3} a^{2^m}, \\
&b^{(m)} = u_m(a,\gamma) b, \;\; u_m(a,\gamma) \equiv \prod_{l=0}^{m-1} u(a^{(l)}, \gamma^{(l)}), \quad \gamma^{(m)} = \gamma^{(-1)^m}, \\
&\alpha^{(m)} = \frac{\gamma^{(m)}(1-b^{(m)})}{1+\gamma^{(m)}}, \;\; \beta^{(m)} = \frac{b^{(m)} + \gamma^{(m)}}{1+\gamma^{(m)}},
\end{aligned} \tag{3·2}$$

where $u_0(a,\gamma) \equiv 1, a^{(0)} \equiv a, b^{(0)} \equiv b$ and $\gamma^{(0)} \equiv \gamma$.

Since $a^{(m)} = (\gamma a)^{2^m} \gamma^{-2[2^{m-1}+(-1)^m]/3}$ and $\gamma a_1 \geq 1$, it is easily shown that $a^{(m)} > 1$ holds for $a > a_1$ and $m \geq 0$. Thus, for $m \geq 1$, $u_m(a,\gamma)$ and $b^{(m)}(a,\gamma)$ are defined for $a > a_1$. In the below, we assume $a > 1$. For $m \geq 1$, $b^{(m)}$ is a continuous strictly decreasing function w.r.t. a for $a > a_1$, and $\lim_{a\to\infty} b^{(m)}(a,\gamma) = b$.

Let us prove the following lemma.

lemma 3'

For integer $m \geq 2$, there exists the unique value of $a = a_m$ greater than 1 such that

$$a^{(m-1)} = \frac{\beta^{(m-1)}}{\alpha^{(m-1)}}, \tag{3·3}$$

which is equivalent to

$$b^{(m)}(a_m) = 1. \tag{3·4}$$

$\{a_m\}_{m=1}^{\infty}$ is the increasing sequence, $1 < a_1 < a_2 < \cdots$, and for $a_m < a, b < b^{(m)} < 1$ for $m \geq 1$.

Proof

The following equivalence relations are easily proved if these quantities are defined.

$$b^{(m)}(a) = 1 \Longleftrightarrow x_M^{(m-1)} = \beta^{(m-1)} \Longleftrightarrow a^{(m-1)} = \frac{\beta^{(m-1)}}{\alpha^{(m-1)}} > 1. \tag{3·5}$$

Let us consider the $m = 2$ case. For $a > 1$, $b^{(1)}$ is defined and a_2 is the solution of the equation $a^{(1)} = \frac{\beta^{(1)}}{\alpha^{(1)}}$. There is the unique solution greater than 1 of this equation, $a_2 = \frac{1}{2\alpha}(1 + \sqrt{1 - \frac{4\alpha^2}{\gamma}})$. $a_2 > a_1$ is easily proved. From the relation (3·4), $b^{(2)} = 1$ at $a = a_2$. Next, let us assume that $b^{(m)}(a_m) = 1$ for $m(\geq 2)$ and $a = a_m > 1$. Then for $a > a_m$, $b < b^{(m)} < 1$ and the function $\beta^{(m)}/\alpha^{(m)} = 1 + \frac{1+\gamma^{(m)}}{\gamma^{(m)}} \frac{b^{(m)}}{1-b^{(m)}}$ is continuous and decreasing

w.r.t. a. On the other hand, $a^{(m)}$ is the continuous increasing function w.r.t. a. Further, we obtain following limits.

$$\lim_{a \to a_m+0} \frac{\beta^{(m)}}{\alpha^{(m)}} = \infty, \quad \lim_{a \to \infty} \frac{\beta^{(m)}}{\alpha^{(m)}} = 1 + \frac{1+\gamma^{(m)}}{\gamma^{(m)}} \frac{b}{1-b} = \text{ finite},$$
$$\lim_{a \to a_m+0} a^{(m)} = \text{ finite}, \quad \lim_{a \to \infty} a^{(m)} = \infty.$$

Thus, there exists the unique value of $a_{m+1} (> a_m)$ such that $a^{(m)} = \beta^{(m)}/\alpha^{(m)} > 1$. Thus, $b^{(m+1)}(a_{m+1}) = 1$. Therefore, the first half of the lemma is proved. The second half immediately follows from the decreasing property of $b^{(m)}$ and the facts $b^{(1)}(a = \beta/\alpha) = 1, a_1 \geq \beta/\alpha$ and $1 < a_1 < a_2$. Q.E.D.

For positive integer m, we define $A_{\boldsymbol{\varphi}^m(\boldsymbol{a})}(x^{(m)})$ for $a > a_m$ from $A_{\boldsymbol{\varphi}^{m-1}(\boldsymbol{a})}(x^{(m-1)})$ successively by the same procedure as in the lemma2'.
We define $x_M^{(m)} \equiv a^{(m)}\alpha^{(m)}$, which is the maximum value of $A_{\boldsymbol{\varphi}^m(\boldsymbol{a})}(x^{(m)})$. Further, we define $x^{(m)*} \equiv \frac{\gamma^{(m)}a^{(m)}}{\gamma^{(m)}a^{(m)}+1}$. $x_M^{(0)} = x_M$ and $x^{(0)*} = x^*$.

lemma 4'

For non-negative integer m and for $a_m < a$ there exists unique non-zero fixed point for $A_{\boldsymbol{\varphi}^m(\boldsymbol{a})}(x^{(m)})$. For $a_m < a < a_{m+1}$, the fixed point is stable. For $m \geq 1$, the stable fixed point is $x_M^{(m)}$ and $\alpha^{(m)} < x_M^{(m)} < \beta^{(m)}$.* At $a = a_{m+1}$, $x_M^{(m)} = x^{(m)*} = \beta^{(m)}$. For $a_{m+1} < a$, the non-zero unstable fixed point is $x^{(m)*}$ and unstable and $x^{(m)*} > \beta^{(m)}$. Here, we define $a_0 \equiv 1, \alpha^{(0)} \equiv \alpha, \beta^{(0)} \equiv \beta$.

Proof

First of all, we notice that $A_{\boldsymbol{\varphi}^m(\boldsymbol{a})}(x^{(m)})$ is really a trapezoid map for $m \geq 1$ because from lemma 3' $b < b^{(m)} < 1$ for $a_m < a$ and $m \geq 1$.

For the case $m = 0$, the statement follows from lemma 1'.

Let us assume that the statement holds for $m \geq 0$. Since $x^{(m)*} = \frac{\gamma^{(m)}a^{(m)}}{\gamma^{(m)}a^{(m)}+1}$ is the unstable fixed point of $A_{\boldsymbol{a}^{(m)}}(x^{(m)})$ for $a_{m+1} < a$, from lemma 2', taking $A^2_{\boldsymbol{a}^{(m)}}(x^{(m)})$ and rescaling $x^{(m)}$ as $x^{(m+1)} = \frac{x^{(m)*} - x^{(m)}}{\Delta x^{(m)}}$, we obtain $A_{\boldsymbol{a}^{(m+1)}}(x^{(m+1)})$ which is defined in $[0, 1]$, where $\Delta x^{(m)} = \gamma^{(m)}(a^{(m)} - 1)/(1 + \gamma^{(m)}a^{(m)})$. From the relation (2·14) it follows that $x_M^{(m)} = \beta^{(m)}$ at $a = a_{m+1}$. Let us consider $x_M^{(m+1)} = a^{(m+1)}\alpha^{(m+1)}$. Since $x_M^{(m+1)}$ is the continuous increasing function w.r.t. a, we obtain $\alpha^{(m+1)} < x_M^{(m+1)} < \beta^{(m+1)}$ for $a_{m+1} < a < a_{m+2}$. This implies $x_M^{(m+1)}$ is the unique fixed point for $A_{\boldsymbol{\varphi}^{m+1}(\boldsymbol{a})}(x^{(m+1)})$ and stable. It is easily shown that for $a_{m+2} < a$, $x_M^{(m+1)}$ is no more fixed point but $x^{(m+1)*} = \frac{\gamma^{(m+1)}a^{(m+1)}}{\gamma^{(m+1)}a^{(m+1)}+1}$ becomes unstable fixed point. Further, it is easily shown that $x^{(m+1)*} > \beta^{(m+1)}$ for $a_{m+1} < a$, and $x_M^{(m+1)} = x^{(m+1)*}$

* For $m = 0$, see lemma 1'

at $a = a_{m+2}$. Q.E.D.

lemma 5'

The unique non-zero fixed point of $A_{\boldsymbol{\varphi}^m(\boldsymbol{a})}(x^{(m)})$ in lemma 4' is the periodic cycle with prime period 2^m.

The proof is the same as that in the symmetric case. Thus, we obtain the following theorem.

Theorem 2

$x = 0$ is period 1 solution of $A_{\boldsymbol{a}}(x)$ for any $a > 0$. Except for this, for any non-negative integer m, for $a_m < a < a_{m+1}$ there is the unique cycle with prime period 2^k for $k = 0, 1, \cdots, m$ and no other periodic points exist. The periodic solution with prime period 2^m is stable for $a_m < a < a_{m+1}$ and unstable for $a_{m+1} < a$. At $a = a_{m+1}$, the period 2^m cycle and the period 2^{m+1} cycle coincide.

The uniqueness of the cycle with prime period 2^k $k = 0, 1, 2, \cdots, m$ for $a_m < a < a_{m+1}$ is proved by the same argument as that in the symmetric case. Thus, likewise the symmetric case, we conclude that as a is increased the period doubling bifurcation cascades, and at $a = a_m$ the periodic solution with prime period 2^m appears.

Now, let us investigate the upper bound of the sequence $\{a_m\}$. For $m \geq 1$ $x_M^{(m)}(a)$ is defined at least for $a_1 < a$. These are the strictly increasing functions w.r.t a. For any $m \geq 1$, $x_M^{(m)}(a_m) = 0$ and $\lim_{a\to\infty} x_M^{(m)}(a) = \infty$ hold. Therefore, for $m \geq 1$, we define $a_M(m)$ as the unique solution of the equation

$$x_M^{(m)}(a) = 1. \tag{3·6}$$

For $m = 0$, we define $a_M(0) = \frac{1}{\alpha}$ and $a_M \equiv a_M(0)$. Then, we obtain the following lemma.

lemma 6'

$$\begin{aligned} 1 < a_1 < a_2 < \cdots < a_m < a_{m+1} < \cdots \\ < a_M(m+1) < a_M(m) \cdots < a_M(2) < a_M(1) < a_M \end{aligned} \tag{3·7}$$

Proof

For $m \geq 1$, since $x_M^{(m)}(a_m) = 0, a_m < a_M(m)$ follows. For $m \geq 0$, $x_M^{(m+1)}$ is rewritten as

$$x_M^{(m+1)} = \gamma^{(m)}\{a^{(m)}\}^2 \frac{1}{a^{(m)} - 1}(x_M^{(m)} - \beta^{(m)}). \tag{3·8}$$

Then, we obtain $x_M^{(m+1)}(a_M(m)) = \frac{a^{(m)}}{a^{(m)}-1} > 1$ for $m \geq 0$. Therefore, $a_M(m+1) < a_M(m)$ for $m \geq 0$. Thus, we obtain $a_m < a_{m+1} < a_M(m+1) < a_M(m)$. Q.E.D.

From this,

$$a_c \equiv \lim_{m\to\infty} a_m \leq \lim_{m\to\infty} a_M(m) < a_M \tag{3·9}$$

follows. Therefore a_c is finite.

Now, let us investigate the symbolic sequence. The sequence of R and L for the period 2^m solution becomes Metropolis-Stein-Stein(MSS) sequence. Let us prove this.

As in the case of the symmetric map, for $m \geq 0$ we define the onset of the 2^n-cycle for $A_{\boldsymbol{\varphi}^m(\boldsymbol{a})}(x^{(m)})$ as $a_n^{(m)}$ and the i-th orbit of the period 2^n cycle as $x_{n,i}^{(m)}$. $a_n^{(0)}$ is equal to previously defined a_n. For $m = 0$, we often omit the superscript (0). As $x_{m,1}$ we set the largest x value among 2^m members of the periodic cycle. Then, $x_{m,1} = x_M = a\alpha$ when the cycle is stable. Further, the coordinate space $[0,1]$ of $x^{(m)}$ into the three intervals as $I_L^{(m)} \equiv [0, \alpha^{(m)}], I_C^{(m)} \equiv (\alpha^{(m)}, \beta^{(m)})$ and $I_R^{(m)} \equiv [\beta^{(m)}, 1]$.

Then, we obtain the corresponding lemma 7', 8' and 9' for $A_{\boldsymbol{a}}(x)$ to the lemma 7, 8, and 9 for $T_{\boldsymbol{a}}(x)$, respectively. We omit the statements and proofs of these lemmas, since they are almost the same as those for $T_{\boldsymbol{a}}(x)$. Thus, we obtain the MSS sequence for the period 2^m solution.

Now, as in the symmetric case, we derive the equations for a_m and a_c, and obtain the asymptotic expression for δ_m. Defining $s(x)$ as before, $f_s(x)$ for $x \in I_L \cup I_R$ is defined as

$$f_s(x) = a(\eta + \nu x), \quad \eta = \gamma s(x), \quad \nu = 1 - (1+\gamma)s(x). \tag{3·10}$$

In the below, s_i, η_i and ν_i are values evaluated at $x = x_{m,i}$. Starting from $x_{m,1} = \alpha a$, we obtain $x_{m,2^m}$ for $a_m \leq a \leq a_{m+1}$,

$$\begin{aligned}
x_{m,2^m} &= f_{s_{2^m-1}}(x_{m,2^m-1}) \equiv \sum_{l=0}^{2^m-1} c_l^{(m)} a^{2^m-l} \equiv F_m(a), \qquad (3·11)\\
c_l^{(m)} &= \eta_l \prod_{j=l+1}^{2^m-1} \nu_j \ \text{ for } \ 1 \leq l \leq 2^m-2,\\
c_0^{(m)} &= \alpha \prod_{j=1}^{2^m-1} \nu_j = \eta_0 \prod_{j=1}^{2^m-1} \nu_j,\\
c_{2^m-1}^{(m)} &= \eta_{2^m-1},
\end{aligned}$$

where $\eta_0 \equiv \alpha$. Thus,

$$F_m(a_m) = \frac{\alpha + \beta + (-1)^{m-1} b}{2}, \tag{3·12}$$

$$F_{m-1}(a_m) = \frac{\alpha + \beta + (-1)^{m-1} b}{2}. \tag{3·13}$$

We define ζ_l and $\hat{\zeta}_l$ as the numbers of 1 in $s_1, s_2, \cdots, s_{2^l-1}$ and in $s_1, s_2, \cdots, s_{2^l}$, respectively [5]. Then, their expressions are

$$\begin{aligned}\hat{\zeta}_l &= (2^{l+1} + (-1)^l)/3 \;\; \text{for } l \geq 0, \\ \zeta_l &= \hat{\zeta}_l - \frac{1+(-1)^l}{2}(l \geq 0), \\ \zeta_l &= \zeta_{l-1} + \hat{\zeta}_{l-1} \;\; \text{for } l \geq 1,\end{aligned} \tag{3·14}$$

where $\zeta_0 \equiv 0$. The following relations hold,

$$\begin{aligned}&\sum_{l=0}^{m-2} \hat{\zeta}_l = \zeta_{m-1}, \\ &\zeta_{2n+1} = \frac{4^{n+1}-1}{3} = \hat{\zeta}_{2n+1}(n \geq 0) \;\; : \text{odd number}, \\ &\zeta_{2n} = \frac{2(4^n-1)}{3} = \hat{\zeta}_{2n} - 1(n \geq 1) \;\; : \text{even number}, \\ &\prod_{j=1}^{2^m-1} \nu_j = (-\gamma)^{\zeta_m} \quad \text{for } m \geq 1.\end{aligned} \tag{3·15}$$

From these it follows that $\hat{\zeta}_l$ is odd number for $l \geq 1$. Then, from the last relation in eq.(3·15), we obtain for $l \geq 0$

$$r_l \equiv (-\gamma)^{-\zeta_m} c_l^{(m)} = \{\prod_{j=1}^{2^m-1} \nu_j\}^{-1} \eta_l \prod_{j=l+1}^{2^m-1} \nu_j = \eta_l \{\prod_{j=1}^{l} \nu_j\}^{-1}.$$

That is, r_l is m-independent as long as it is defined. Thus, we get

$$r_l = \eta_l \{\prod_{j=1}^{l} \nu_j\}^{-1} \text{ for any } l(\geq 0), \tag{3·16}$$

$$r_0 = \alpha, \;\; r_{2^m} = -s_{2^m} \gamma^{1-\hat{\zeta}_m} \;(m \geq 0). \tag{3·17}$$

Thus, we define

$$G_m(a) \equiv (-\gamma)^{-\zeta_m} a^{-2^m} F_m(a) = \sum_{l=0}^{2^m-1} r_l a^{-l}. \tag{3·18}$$

Let us consider the condition for the convergence of $G_m(a)$. Using the relations (3·19) and

$$r_l = -\gamma^{\hat{\zeta}_m} r_{2^m+l} \text{ for } 1 \leq l \leq 2^m - 1 \;\; (m \geq 1), \tag{3·19}$$

we obtain the following recursive relation for $m \geq 1$,

$$G_{m+1}(a) = (1 - \tilde{a}_m) G_m(a) + \tilde{a}_m (\alpha - \gamma s_{2^m}), \tag{3·20}$$

and then we obtain for $m \geq 1$

$$G_m(a) = \alpha \prod_{l=0}^{m-1}(1-\tilde{a}_l) + \sum_{k=0}^{m-1}[\prod_{l=k+1}^{m-1}(1-\tilde{a}_l)]\tilde{a}_k(\alpha - \gamma s_{2^k}), \tag{3·21}$$

where $\tilde{a}_l = a^{-2^l}\gamma^{-\hat{\zeta}_l} = \frac{1}{\sqrt{a^{(l-1)}}}$. Since $\tilde{a}_l = (a\gamma^{2/3})^{-2^l}\gamma^{-(-1)^l/3}$, putting $\hat{\gamma} = \max\,(\gamma, \gamma^{-1})$, we get

$$\tilde{a}_l \leq (a\gamma^{2/3})^{-2^l}\hat{\gamma}^{1/3}. \tag{3·22}$$

Thus, if $a\gamma^{2/3} > 1, \lim_{m\to\infty} G_m(a)$ converges. That is, $G_\infty(z) \equiv \lim_{m\to\infty} G_m(z)$ is the analytic function for $|z| > \gamma^{-2/3}$. * From eq.(3 · 12) we obtain

$$G_m(a_m) = (-\gamma)^{-\zeta_m} a_m^{-2^m} \frac{\alpha + \beta + (-1)^{m-1}b}{2},^* \tag{3·23}$$

and then

$$G_\infty(a_c) = 0.^* \tag{3·24}$$

Similar to the symmetric case, we obtain from eq.(3·23)

$$\epsilon_m = \{G_m(a_c) - \mathit{\Gamma} a_c^{-2^m}\}(1+\bar{h}_m)/G'_\infty(a_c), \tag{3·25}$$
$$\mathit{\Gamma} = (-1)^{-\zeta_m}\frac{\alpha+\beta+(-1)^{m-1}b}{2}, \quad \bar{h}_m = \frac{G'_\infty(a_c)}{G'_m(\bar{a}_m) + \mathit{\Gamma} 2^m(\hat{a}_m)^{-2^m-1}} - 1,$$

where $a_m < \bar{a}_m < a_c, \;\; a_m < \hat{a}_m < a_c$ and $\lim_{m\to\infty}\bar{h}_m = 0$. Further, using the relation (3·17) we get

$$G_m(a_c) = -a_c^{-2^m}\gamma^{-\hat{\zeta}_m}(\alpha - \gamma s_{2^m}) + (a_c^{-2^m}\gamma^{-\hat{\zeta}_m})^2 q_m, \tag{3·26}$$
$$q_m = -\frac{1}{1-a_c^{-2^m}\gamma^{-\hat{\zeta}_m}}(\alpha - \gamma s_{2^m} + \gamma^{2\hat{\zeta}_m}\sum_{l=0}^{\infty} r_{2^{m+1}+l}a_c^{-l}).$$

Substituting (3·26) into (3·25) , we obtain

$$\epsilon_m = a_c - a_m = \frac{b a_c^{-2^m}\gamma^{-\zeta_m}}{G'_\infty(a_c)}(1+h_m), \tag{3·27}$$
$$h_m = \bar{h}_m(1 + a_c^{-2^m}\gamma^{\zeta_m - 2\hat{\zeta}_m}\frac{q_m}{b}) + a_c^{-2^m}\gamma^{\zeta_m-2\hat{\zeta}_m}\frac{q_m}{b}. \tag{3·28}$$

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* In the previous paper [5], as the condition for the convergence of $G_\infty(z)$ we gave $|z| > \max\,(1, \gamma^{-1})$. This is a sufficient condition.

* This is the equation $(3 \cdot 4)$ in ref. [5]. There are several misprints in [5]. ζ_m in the definition of $G_m(a)$ and eq.$(3 \cdot 4)$ should be changed to $-\zeta_m$.

* Note that $a_c\gamma^{2/3} > 1$ follows from $a_1\gamma > 1$. Since $a_c^{-2^m}\gamma^{-\hat{\zeta}_m} = \tilde{a}_m(a = a_c) = (a_c\gamma^{2/3})^{-2^m}\gamma^{-(-1)^m/3}$ and $a_c^{-2^m}\gamma^{-\zeta_m} = a_c^{-2^m}\gamma^{-\hat{\zeta}_m}\gamma^{\frac{1+(-1)^m}{2}}$ these tend to 0 as $m \to \infty$.

$\lim_{m\to\infty} h_m = 0$ is shown in appendix B. Thus,

$$\begin{aligned}\delta_m &= \gamma^{(-1)^m/3}(a_c\gamma^{2/3})^{2^m}(1+l_m), \\ &\quad l_m = [h_m - h_{m+1} - a_c^{-2^m}\gamma^{-\hat{\zeta}_m}(1+h_{m+1}) + a_c^{-2^{m+1}}\gamma^{-\hat{\zeta}_{m+1}}(1+h_{m+2})] \\ &\quad /[1+h_{m+1} - a_c^{-2^{m+1}}\gamma^{-\hat{\zeta}_{m+1}}(1+h_{m+2})],\end{aligned} \tag{3·29}$$

and $\lim_{m\to\infty} l_m = 0$.

To obtain a similar relation to the relation (2·32) in the case of symmetric map, we define $v_m(a,\gamma)$ as

$$v_m(a,\gamma) = 1/u_m(a,\gamma) = \prod_{l=0}^{m-1} v(a^{(l)},\gamma^{(l)}), \quad v(a,\gamma) = \frac{1}{u(a,\gamma)} = \frac{\gamma(a-1)}{\gamma a+1}. \tag{3·30}$$

For $m \geq 1$, $v_m(a)$ is rewritten as

$$\begin{aligned}v_m(a) &= \frac{(1-a^{-1})}{g_{m-1}(a)}\prod_{l=0}^{m-2} h_l(a), \\ &\quad g_l(a) \equiv 1+\gamma^{-\hat{\zeta}_l}a^{-2^l}, \quad h_l(a) \equiv 1-\gamma^{-\hat{\zeta}_l}a^{-2^l}, \text{ for } l \geq 0, \\ &\quad \prod_{l=0}^{-1} h_l(a) \equiv 1, \quad v_1(a) = v(a).\end{aligned} \tag{3·31}$$

Then, for $m \geq 2$, the following equation is proved by using the above relations

$$G_{m-1}(a) = \alpha + \frac{1}{1+\gamma^{-1}}\{g_{m-1}(a)v_m(a) - 1 - (-1)^m a^{-2^{m-1}}\gamma^{-\zeta_{m-1}}\}. \tag{3·32}$$

See Appendix A. Thus, the condition for a_m, $v_m(a_m) = b$, is equivalent to the following equation derived from eq.(3·13),

$$G_{m-1}(a_m) = (-\gamma)^{-\zeta_{m-1}} a_m^{-2^{m-1}} \frac{\alpha+\beta+(-1)^{m-1}b}{2}. \tag{3·33}$$

From relation (3·32), we get for $a > \gamma^{-2/3}$

$$G_\infty(a) = \alpha + \frac{1}{1+\gamma^{-1}}(v_\infty(a) - 1). \tag{3·34}$$

Thus,

$$\begin{aligned}G'_\infty(a) &= \frac{1}{1+\gamma^{-1}}v'_\infty(a) = \frac{2b}{1+\gamma^{-1}}\tau(a), \\ &\quad \tau(a) \equiv \sum_{l=0}^{\infty} \frac{1+\gamma^{(l)}}{2(a^{(l)}-1)(\gamma^{(l)}a^{(l)}+1)}\gamma^{2(2^l-(-1)^l)/3}2^l a^{2^l-1} > 0.\end{aligned} \tag{3·35}$$

Therefore, we obtain

$$\epsilon_m = \frac{1}{2\tau(a_c)} a_c^{-2^m}\gamma^{-\zeta_m}(1+\gamma^{-1})(1+h_m). \tag{3·36}$$

§4. Period doubling bifurcation of the period p solution : symmetric case

In this section, we consider the period doubling bifurcation of the periodic solution with the prime period $p(\geq 3)$ which emerges by a tangent bifurcation.

Let us consider the mapping $T^p_{\boldsymbol{a}}(x)$. At $a = a_M = \frac{2}{1-b}$, the map looks like Fig.5.

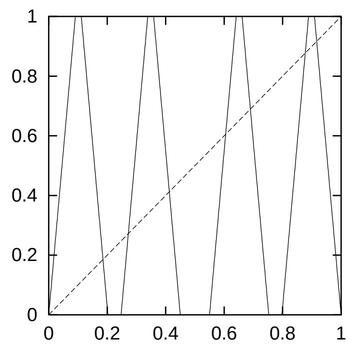

Fig. 5. Trapezoid map $T^3_{(a,b)}(x)$ for $a = a_M$ with $b = 0.1$.

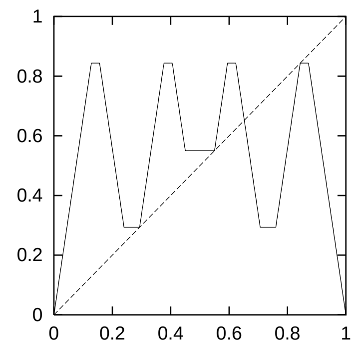

Fig. 6. Trapezoid map $T^3_{(a,b)}(x)$ for $a = a_{p,0}$ with $b = 0.1$.

As a is decreased from $a = a_M$, at some value of $a = a_{p,0}$, the map becomes tangent to the line $y = x$, as is illustrated in Fig.6. If we put $x_1 = a\frac{1-b}{2}$, at this point the symbols for $x_1, x_2, \cdots, x_{p-1}$ are $RL\cdots L$. Then,

$$x_p = a^{p-1}(1 - \frac{a(1-b)}{2}). \tag{4·1}$$

As is shown in Appendix C, $x_p(a)$ takes the maximum value at $a = a_{p,max} \equiv \frac{p-1}{p} a_M$ and $\frac{1+b}{2} < x_p(a_{p,max})$ for $p \geq 3$. Thus, the point $a_{p,0}$ where the tangent bifurcation takes place satisfies the condition

$$x_p(a_{p,0}) = \frac{1+b}{2}. \tag{4·2}$$

This equation has two solutions, one is $a = 1$ and the other corresponds to $a_{p,0}$. Thus, we obtain

$$1 < \frac{p-1}{p} a_M < a_{p,0} < a_{p,1},$$

where $a_{p,1}$ is defined by

$$x_p(a_{p,1}) = \frac{1-b}{2}, \tag{4·3}$$

and $a_{p,1} > 1$. See Fig.7.

$a_{p,1}$ is the onset point of the period $2p$ point. The symbolic sequence for $x_1, x_2, \cdots, x_{p-1}$ is $RL\cdots L$ for $a_{p,0} < a$. See Appendix D. For $a_{p,0} < a < a_{p,1}$, let us consider the map $T^p_{\boldsymbol{a}}(x)$. The unstable periodic point with period p of $T_{\boldsymbol{a}}(x)$ is

$$x^* = \frac{a^{p-1}(a-1)}{a^p - 1}. \tag{4·4}$$

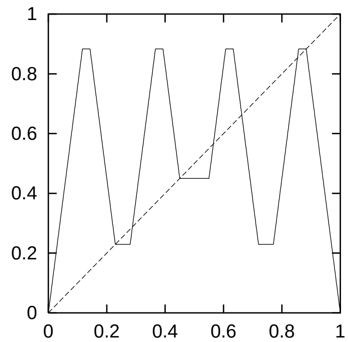

Fig. 7. Trapezoid map $T^3_{(a,b)}(x)$ for $a = a_{p,1}$ with $b = 0.1$.

Rescaling the map $T^p_{\boldsymbol{a}}(x)$ around $x = 1/2$ for $a > a_{p,0}$, we obtain the trapezoid map $T_{\boldsymbol{a}^{(0)}}(x^{(0)})$, where

$$\boldsymbol{a}^{(0)} = (a^{(0)}, b^{(0)}), \quad a^{(0)} = a^p, \quad b^{(0)} = \frac{b}{\Delta x} = r(a)b, \quad \Delta x = 2x^* - 1, \qquad (4\cdot5)$$
$$r(a) = \frac{1}{\Delta x} = \frac{a^p - 1}{a^p - 2a^{p-1} + 1}, \quad x^{(0)} = \frac{x^* - x}{2x^* - 1}.$$

For $a > 1$, $r'(a) < 0$. As before, we define $x^{(0)}_M = a^{(0)}\frac{1-b^{(0)}}{2}$. For $a_{p,0} < a < a_{p,1}$, $x^{(0)}_M$ corresponds to x_p, that is $x^{(0)}_M = \frac{x^*-x_p}{2x^*-1}$. Therefore, at $a = a_{p,0}$, $x^{(0)}_M = 0$ and at $a = a_{p,1}$, $x^{(0)}_M = \frac{1+b^{(0)}}{2}$. We note that $b^{(0)}(a_{p,0}) = 1$. Since $r(a)$ is the strictly decreasing continuous function w.r.t. a for $a > 1$, $b^{(0)}(a)$ is the strictly decreasing continuous function and $x^{(0)}_M(a)$ is the strictly increasing continuous function w.r.t. a. Therefore, for $a_{p,0} < a < a_{p,1}$, $\frac{1-b^{(0)}}{2} < x^{(0)}_M < \frac{1+b^{(0)}}{2}$, and for $a_{p,1} < a$, $\frac{1+b^{(0)}}{2} < x^{(0)}_M$. That is, $x^{(0)}_M$ is the stable fixed point of $T_{\boldsymbol{a}^{(0)}}(x^{(0)})$ for $a_{p,0} < a < a_{p,1}$, and for $a_{p,1} < a$ the fixed point is $x^*_u \equiv \frac{a^{(0)}}{a^{(0)}+1}$ and is unstable. For $a_{p,1} < a$, rescaling the map $T^2_{\boldsymbol{a}^{(0)}}(x^{(0)})$ we obtain the trapezoid map $T_{\boldsymbol{a}^{(1)}}(x^{(1)})$, where,

$$\boldsymbol{a}^{(1)} \equiv \boldsymbol{\varphi}(\boldsymbol{a}^{(0)}) = (a^{(1)}, b^{(1)}), \quad a^{(1)} = \{a^{(0)}\}^2 = a^{2p}, \qquad (4\cdot6)$$
$$b^{(1)} = u(a^{(0)})b^{(0)} = u(a^p)r(a)b, \quad u(a) = \frac{a+1}{a-1}.$$

As before, we define $\boldsymbol{a}^{(m)}$ as

$$\boldsymbol{a}^{(m)} \equiv \boldsymbol{\varphi}^m(\boldsymbol{a}^{(0)}) = (a^{(m)}, b^{(m)}), \quad a^{(m)} = \{a^{(m-1)}\}^2 = a^{2^m p}, \qquad (4\cdot7)$$
$$b^{(m)} = u(a^{(m-1)})b^{(m-1)} = u_m(a)r(a)b, \quad u_m(a) = \prod_{l=0}^{m-1} u(a^{(l)}).$$

Since $b^{(m)}(a)$ is the decreasing function w.r.t. a, we can repeat the same argument as in section 2. At $a = a_{p,m}$, the period 2^m solution of $T_{\boldsymbol{a}^{(0)}}(x^{(0)})$ appears and $a_{p,m}$ satisfies the condition

$$b^{(m)}(a_{p,m}) = 1, \quad \text{for } m \geq 0, \qquad (4\cdot8)$$

and

$$1 < a_{p,0} < a_{p,1} < a_{p,2} < \cdots < \frac{2}{1-b} = a_M. \tag{4·9}$$

The symbolic sequence for the period 2^m solution is the MSS sequence for the map $T_{\boldsymbol{a}^{(0)}}(x^{(0)})$. For $a_{p,m} \leq a \leq a_{p,m+1}$, starting from $x_1^{(0)} = x_M^{(0)}$, $x_{2^m}^{(0)}$ is expressed by the function $F_m(a^{(0)}, b^{(0)})$. Here, $F_m(a,b)$ is defined by eq.(2 · 10) and we include b as an independent variable to F_m explicitly. Thus, we obtain the other condition for $a_{p,m}$,

$$F_m(a^{(0)}(a_{p,m}), b^{(0)}(a_{p,m})) = \frac{1 + (-1)^{m-1} b^{(0)}(a_{p,m})}{2}. \tag{4·10}$$

In terms of $G_m(a,b)$ defined by eq.(2·19), the equation (2·21) becomes

$$G_m(a^{(0)}(a_{p,m}), b^{(0)}(a_{p,m})) = a^{(0)}(a_{p,m})^{-2^m} \frac{(-1)^m - b^{(0)}(a_{p,m})}{2}, \tag{4·11}$$

and we obtain

$$G_\infty(a^{(0)}(a_{p,c}), b^{(0)}(a_{p,c})) = 0, \tag{4·12}$$

where $a_{p,c} = \lim_{m\to\infty} a_{p,m}$. From these equations, we get for $\epsilon_{p,m} = a_{p,c} - a_{p,m}$,

$$\epsilon_{p,m} \simeq \frac{a_{p,c}^{-p2^m} r(a_{p,c}) b}{\frac{\partial G_\infty}{\partial a^{(0)}} p a_{p,c}^{p-1} + \frac{1}{2}|r'(a_{p,c})| b}. \tag{4·13}$$

Since $\frac{\partial G_\infty(a,b)}{\partial a} > 0$, the denominator is positive. Therefore, we obtain

$$\delta_m \sim a_{p,c}^{p2^m}. \tag{4·14}$$

As for the relations among $a_{p,0}$ s, the following ordering holds,

$$a_c \leq a_{3,0} < a_{4,0} < \cdots < a_{p,0} < a_{p+1,0} < \cdots < a_M. \tag{4·15}$$

See Appendix E.

§5. Period doubling bifurcation of the period $p \geq 3$ solution : asymmetric case

As in the previous case, we investigate the period doubling bifurcation of the period $p(\geq 3)$ solution in the asymmetric case.

In this case also, we consider the period p solution starting with $x_1 = a\alpha$ with the sequence of symbols, $RL \cdots L$. Then, we obtain

$$x_p = \gamma a^{p-1}(1 - a\alpha). \tag{5·1}$$

As is shown in Appendix C, $x_p(a)$ takes the maximum value at $a = a_{p,max} \equiv \frac{p-1}{p} a_M$ and $\beta < x_p(a_{p,max})$ for $p \geq 3$, where $a_M = 1/\alpha$ as before. Then, the onset point $a_{p,0}(> a_{p,max})$ of the period p solution satisfies

$$x_p(a_{p,0}) = \beta. \tag{5·2}$$

Although this equation has two solutions, we should adopt the larger one as in the symmetric case. The symbolic sequence for $x_1, x_2, \cdots, x_{p-1}$ is $RL \cdots L$ for $a_{p,0} \leq a$. See Appendix D. On the other hand, the onset point of the period $2p$ solution, $a_{p,1}$, satisfies

$$x_p(a_{p,1}) = \alpha. \tag{5·3}$$

Then, for $a_{p,0} < a$, the unstable periodic point x^* with period p for $A_{\boldsymbol{a}}(x)$ is

$$x^* = \frac{(\gamma a - 1)\gamma a^{p-1}}{\gamma^2 a^p - 1}. \tag{5·4}$$

Similar to the symmetric case, we obtain an asymmetric trapezoid map $A_{\boldsymbol{a}^{(0)}}(x^{(0)})$ by rescaling the map $A^p_{\boldsymbol{a}}(x)$ in the vicinity of I_C for $a > a_{p,0}$, where

$$\begin{aligned}
\boldsymbol{a}^{(0)} &= (a^{(0)}, b^{(0)}, \gamma^{(0)}), \quad a^{(0)} = \gamma^2 a^p, \quad \gamma^{(0)} = \gamma^{-1}, \quad b^{(0)} = \frac{b}{\Delta x} = r(a)b, \\
\Delta x &= \frac{\gamma g(a)}{\gamma^2 a^p - 1}, \quad g(a) = \gamma a^p - (1+\gamma)a^{p-1} + 1, \\
r(a) &= 1/\Delta x = \frac{\gamma^2 a^p - 1}{\gamma g(a)}, \quad x^{(0)} = \frac{x^* - x}{\Delta x}, \\
\alpha^{(0)} &\equiv \frac{x^* - \beta}{\Delta x} = \frac{\gamma^2 a^p (1-\beta) - \gamma a^{p-1} + \beta}{\gamma g(a)}, \\
\beta^{(0)} &\equiv \frac{x^* - \alpha}{\Delta x} = \frac{\gamma^2 a^p (1-\alpha) - \gamma a^{p-1} + \alpha}{\gamma g(a)}.
\end{aligned} \tag{5·5}$$

$r'(a)$ becomes

$$r'(a) = -\frac{(1+\gamma)a^{p-2} f(a)}{\gamma g(a)^2}, \tag{5·6}$$

where $f(a) = \gamma^2 a^p - p\gamma a + p - 1$. We can prove that at least for $a_{p,0} \leq a$, $f(a) > 0$. Thus, for $a_{p,0} \leq a$, $r'(a) < 0$. Further, $\alpha^{(0)\prime}(a) = \frac{ba^{p-2} f(a)}{\gamma g(a)^2} > 0$ and $\beta^{(0)\prime}(a) = -\gamma \alpha^{(0)\prime}(a) < 0$ for $a \geq a_{p,0}$. Thus, $b^{(0)}$ is the strictly decreasing continuous function and $x_M^{(0)} = a^{(0)}\alpha^{(0)}$ is the strictly increasing continuous function w.r.t. a. At $a = a_{p,0}$, $\alpha^{(0)} = 0, \beta^{(0)} = b^{(0)} = 1$. For $a_{p,0} \leq a \leq a_{p,1}$ $x_p(a)$ corresponds to $x_M^{(0)}$, that is, $x_M^{(0)} = \frac{x^* - x_p}{\Delta x}$. Then, at $a = a_{p,0}, x_M^{(0)} = 0$ and at $a = a_{p,1}, x_M^{(0)} = \beta^{(0)}$. Thus, $\alpha^{(0)} < x_M^{(0)} < \beta^{(0)}$ for $a_{p,0} < a < a_{p,1}$, and $x_M^{(0)}$ is the stable fixed point for $A_{\boldsymbol{a}^{(0)}}(x^{(0)})$. Further, for $a_{p,1} < a$, $x_M^{(0)} > \beta^{(0)}$ and the fixed point

becomes $x_u^* \equiv \frac{\gamma^{(0)}a^{(0)}}{1+\gamma^{(0)}a^{(0)}}$ and unstable. For $a_{p,1} < a$, rescaling the map $A^2_{\boldsymbol{a}^{(0)}}(x^{(0)})$, we obtain $A_{\boldsymbol{a}^{(1)}}(x^{(1)})$, where

$$\begin{aligned}
&\boldsymbol{a}^{(1)} \equiv \boldsymbol{\varphi}(\boldsymbol{a}^{(0)}) = (a^{(1)}, b^{(1)}, \gamma^{(1)}), \;\; a^{(1)} = \{\gamma^{(0)}a^{(0)}\}^2 = \gamma^2 a^{2p}, \\
&b^{(1)} = u(a^{(0)}, \gamma^{(0)})b^{(0)} = u(a^p)r(a)b, \;\; \gamma^{(1)} = \{\gamma^{(0)}\}^{-1}, \;\; u(a,\gamma) = \frac{\gamma a + 1}{\gamma(a-1)}.
\end{aligned} \tag{5·7}$$

Thus, $a^{(1)}$ is increasing, $b^{(1)}$ is decreasing and $\gamma^{(1)}$ is constant w.r.t. a. Further, we define $\boldsymbol{a}^{(m)}$ as

$$\begin{aligned}
&\boldsymbol{a}^{(m)} \equiv \boldsymbol{\varphi}^m(\boldsymbol{a}^{(0)}) = (a^{(m)}, b^{(m)}, \gamma^{(m)}), \;\; a^{(m)} = \{\gamma^{(m-1)}a^{(m-1)}\}^2, \\
&b^{(m)} = u(a^{(m-1)}, \gamma^{(m-1)})b^{(m-1)} = u_m(a,\gamma)r(a)b, \;\; u_m(a,\gamma) = \prod_{l=0}^{m-1} u(a^{(l)}, \gamma^{(l)}), \\
&\gamma^{(m)} = \{\gamma^{(m-1)}\}^{-1}.
\end{aligned} \tag{5·8}$$

Since $b^{(m)}$ is the decreasing function w.r.t. a, we can make a quite similar argument to that in §4. In particular, at $a = a_{p,m}$, the period doubling bifurcation takes place and

$$b^{(m)}(a_{p,m}) = 1 \;\text{ for } m \geq 0, \tag{5·9}$$

and

$$1 < a_{p,0} < a_{p,1} < a_{p,2} < \cdots < \frac{1}{\alpha} = a_M. \tag{5·10}$$

The symbolic sequence for the period 2^m solution is the MSS sequence for the map $A_{\boldsymbol{a}^{(0)}}(x^{(0)})$. For $a_{p,m} \leq a \leq a_{p,m+1}$, starting from $x_1^{(0)} = x_M^{(0)}$, $x_{2^m}^{(0)}$ is expressed by the function $F_m(a^{(0)}, b^{(0)}, \gamma^{(0)})$ and we get

$$F_m(a^{(0)}(a_{p,m}), b^{(0)}(a_{p,m}), \gamma^{(0)}) = \frac{\alpha^{(0)}(a_{p,m}) + \beta^{(0)}(a_{p,m}) + (-1)^{m-1}b^{(0)}(a_{p,m})}{2}, \tag{5·11}$$

where $F_m(a, b, \gamma)$ is defined by eq.(3·11). In terms of $G_m(a, b, \gamma)$ defined by eq.(3·18), the equation (5·11) becomes

$$\begin{aligned}
&G_m(a^{(0)}(a_{p,m}), b^{(0)}(a_{p,m}), \gamma^{(0)}) = \\
&(-\gamma^{(0)})^{-\zeta_m}\{a_m^{(0)}(a_{p,m})\}^{-2^m} \frac{\alpha^{(0)}(a_{p,m}) + \beta^{(0)}(a_{p,m}) + (-1)^{m-1}b^{(0)}(a_{p,m})}{2}.
\end{aligned} \tag{5·12}$$

Let us define $a_{p,c} = \lim_{m\to\infty} a_{p,m}$. Thus, we obtain

$$G_\infty(a^{(0)}(a_{p,c}), b^{(0)}(a_{p,c}), \gamma^{(0)}) = 0, \tag{5·13}$$

and $\epsilon_{p,m} = a_{p,c} - a_{p,m}$ is given by

$$\epsilon_{p,m} \simeq \frac{(\gamma^2 a_{p,c}^p)^{-2^m}\gamma^{\zeta_m} r(a_{p,c})b}{\frac{\partial G_\infty}{\partial a^{(0)}}\gamma^2 p a_{p,c}^{p-1} + \frac{1}{2}|r'(a_{p,c})|b}, \tag{5·14}$$

and then

$$\delta_m \simeq \gamma^{(-1)^{m-1}/3}(a_{p,c}^p \gamma^{4/3})^{2^m}. \tag{5·15}$$

As for the relations among $a_{p,0}$ s, the following ordering holds,

$$a_c \leq a_{3,0} < a_{4,0} < \cdots < a_{p,0} < a_{p+1,0} < \cdots < a_M. \tag{5·16}$$

See Appendix E.

§6. Summary and discussion

In this paper, we studied the symmetric and the asymmetric trapezoid maps rigorously. We gave the proofs of several results about the period doubling bifurcation which occurs as the parameter a is increased. We obtained following scaling results for the period doubling bifurcation starting from a period one solution.

1. Symmetric case

$$\begin{aligned} \epsilon_m &= \frac{b a_c^{-2^m}}{G'_\infty(a_c)}(1+h_m), \\ \delta_m &= a_c^{2^m}(1+l_m) \end{aligned}$$

   where $\lim_{m\to\infty} h_m = 0$ and $\lim_{m\to\infty} l_m = 0$.

2. Asymmetric case

$$\begin{aligned} \epsilon_m &= \frac{b a_c^{-2^m} \gamma^{-\zeta_m}}{G'_\infty(a_c)}(1+h_m), \\ \delta_m &= \gamma^{(-1)^m/3}(a_c \gamma^{2/3})^{2^m}(1+l_m), \end{aligned}$$

   where $\lim_{m\to\infty} h_m = 0$ and $\lim_{m\to\infty} l_m = 0$.

The new results in this paper are on the period doubling bifurcation which starts from the periodic solution with any prime period $p(\geq 3)$.

1. Symmetric case

$$\begin{aligned} \epsilon_{p,m} &\simeq \frac{a_{p,c}^{-p2^m} r(a_{p,c}) b}{\frac{\partial G_\infty}{\partial a^{(0)}} p a_{p,c}^{p-1} + \frac{1}{2}|r'(a_{p,c})| b}. \\ \delta_m &\simeq a_{p,c}^{p2^m}. \end{aligned}$$

2. Asymmetric case

$$\epsilon_m \simeq \frac{(\gamma^2 a_{p,c}^p)^{-2^m}\gamma^{\zeta_m} r(a_{p,c})b}{\frac{\partial G_\infty}{\partial a^{(0)}}\gamma^2 p a_{p,c}^{p-1} + \frac{1}{2}|r'(a_{p,c})|b},$$
$$\delta_m \simeq \gamma^{(-1)^{m-1}/3}(a_{p,c}^p \gamma^{4/3})^{2^m}.$$

These results imply that for any period doubling cascade, the accumulation rate to the accumulation point is extremely fast, in fact, it is exponential.

In a one-dimensional map with one hump, the Feigenbaum constant depends on the power z which characterizes the behaviour of the map in the vicinity of the critical point. It has been proved that $\lim_{z\to\infty}\delta(z) = \text{finite}$ [4]. The result in this paper shows that $\delta(\infty)$ is infinity. That is, the superconvergence takes place only in the case $z = \infty$.
This feature is considered to be attributed to the flatness of the summit. For example, in a one humped map with a flat summit, superconvergence will take place if it satisfies some conditions, e.g. the absolute value of the derivative is greater than some constant $\lambda > 1$ outside a region which contains the flat part of the map.

In the other papers [7), 8), 9)], similar results were obtained by a different method. In those studies, the authors estimated quantities in question by inequalities and obtained that a_m is quadratically convergent, in the case of period doubling of period one solution. This implies $\lim_{m\to\infty}\frac{\ln\delta_m}{\ln\delta_{m-1}} = 2$. On the other hand, our method is constructive. In fact, we gave the precise equation for the onset point a_m of the periodic point with period 2^m and also gave the equation for the accumulation point a_c. From these equations, we obtained the above scaling relations. We also extended the argument not only to the asymmetric trapezoid map but also period doubling of the prime period $p(\geq 3)$ solution which emerges from a tangent bifurcation.

In [8)], the authors mention the problem of convergence of the sequence $\varDelta\epsilon_m/(\varDelta\epsilon_{m-1})^2$, and from our result, this is $\epsilon_m/\epsilon_{m-1}^2 \simeq G'_\infty(a_c)\gamma^{(-1+(-1)^m)/2}/b$ and does not converge for the asymmetric case.

The trapezoid maps studied here are a kind of exactly solvable models of the renomalization group. It is interesting to investigate the further detailed bifurcation structures in these models. This is a future problem.

Appendix A
—— *Proof of relations* (2 · 32) *and* (3 · 32) ——

In this appendix, we prove the relation (3·32). The relation (2·32) is obtained by putting $\gamma = 1$.

For $m \geq 2$, the relation $(3 \cdot 32)$ is

$$G_{m-1}(a) = \alpha + \frac{1}{1+\gamma^{-1}}\{g_{m-1}(a)v_m(a) - 1 - (-1)^m a^{-2^{m-1}}\gamma^{-\zeta_{m-1}}\}, \qquad \text{(A·1)}$$

$$v_m(a) = \frac{(1-a^{-1})}{g_{m-1}(a)}\prod_{l=0}^{m-2} h_l(a),$$

$$g_l(a) \equiv 1 + \gamma^{-\hat{\zeta}_l}a^{-2^l}, \quad h_l(a) \equiv 1 - \gamma^{-\hat{\zeta}_l}a^{-2^l}, \text{ for } l \geq 0.$$

Let us define ϕ_m for $m \geq 1$ as

$$\phi_m(a) \equiv g_m(a)v_{m+1}(a) = (1-a^{-1})\prod_{l=0}^{m-1} h_l(a). \qquad \text{(A·2)}$$

We expand $\phi_m(a)$ as

$$\phi_m(a) \equiv \sum_{l=0}^{2^m} w_l^{(m)} a^{-l}, \qquad \text{(A·3)}$$

and then using relations (3·15) we obtain for $m \geq 1$,

$$w_0^{(m)} = 1, \quad w_{2^m}^{(m)} = (-1)^{m-1}\gamma^{-\zeta_m}. \qquad \text{(A·4)}$$

In particular,

$$w_1^{(1)} = -(1+\gamma^{-1}). \qquad \text{(A·5)}$$

Let us derive the recursive relations for $w_l^{(m)}$. For $m \geq 1$,

$$\phi_{m+1}(a) = g_m(a)v_{m+1}(a) = h_m(a)\phi_m(a) = \phi_m(a) - \gamma^{-\hat{\zeta}_m}a^{-2^m}\phi_m(a). \qquad \text{(A·6)}$$

Comparing coefficients, we obtain for $m \geq 1$,

$$\mathcal{O}(a^{-2^{m+1}}): \quad w_{2^{m+1}}^{(m+1)} = -\gamma^{-\hat{\zeta}_m} w_{2^m}^{(m)}, \qquad \text{(A·7)}$$

$$\mathcal{O}(a^{-2^m}): \quad w_{2^m}^{(m+1)} = -w_{2^m}^{(m)} - \gamma^{-\hat{\zeta}_m} w_0^{(m)}, \qquad \text{(A·8)}$$

$$\mathcal{O}(a^{-(2^m+l)}): \quad w_{2^m+l}^{(m+1)} = -\gamma^{-\hat{\zeta}_m} w_l^{(m)} \text{ for } 0 < l < 2^m, \qquad \text{(A·9)}$$

$$\mathcal{O}(a^{-l}): \quad w_l^{(m+1)} = w_l^{(m)} \text{ for } 0 \leq l < 2^m. \qquad \text{(A·10)}$$

Using the relation (A·4), eq.(A·7) is automatically satisfied. Form eq.(A·8), we get

$$w_{2^m}^{(m+1)} = -(1+\gamma^{-1})s_{2^m}\gamma^{1-\hat{\zeta}_m}. \qquad \text{(A·11)}$$

Now, let us return to the quantity $G_{m-1}(a)$. We put the r.h.s. of relation (A·1) as $\hat{G}_{m-1}(a)$, and expand it by a^{-1} for $m \geq 2$.

$$\hat{G}_{m-1}(a) = \sum_{l=0}^{2^{m-1}-1} \bar{r}_l^{(m-1)} a^{-l}. \qquad \text{(A·12)}$$

Thus, we obtain the following relations for $m \geq 2$,

$$\bar{r}_0^{(m-1)} = \alpha, \tag{A$\cdot$13}$$

$$\bar{r}_l^{(m-1)} = \frac{w_l^{(m-1)}}{1+\gamma^{-1}}, \quad \text{for } 0 < l \leq 2^{m-1}-1. \tag{A$\cdot$14}$$

Thus, from (A·11), (A·9) and (A·10) we obtain for $m \geq 1$

$$\bar{r}_{2^m}^{(m+1)} = -s_{2^m}\gamma^{1-\hat{\zeta}_m}, \tag{A$\cdot$15}$$

$$\bar{r}_{l+2^m}^{(m+1)}, = -\gamma^{-\hat{\zeta}_m}\bar{r}_l^{(m)} \quad \text{for } 0 < l \leq 2^m - 1, \tag{A$\cdot$16}$$

$$\bar{r}_l^{(m+1)} = \bar{r}_l^{(m)} \quad \text{for } 0 \leq l \leq 2^m - 1. \tag{A$\cdot$17}$$

The relation (A·17) implies that $\bar{r}_l^{(m)}$ is m independent as long as it is defined and so we omit the superscript (m). From relations (A·15) and (A·16) it turns out that $\bar{r}_l$ for $l = 2^m+1, \cdots, 2^{m+1}-1$ are determined by $\bar{r}_l$ for $l = 1, \cdots, 2^m - 1$. Thus, $\bar{r}_1$ and $\bar{r}_{2^m}$ for $(m \geq 1)$ determines $\bar{r}_l$ for $l > 0$. From eqs.(A·5) and (A·14) $\bar{r}_1 = -1 = r_1$ follows. Also, from eqs.(3·17) and (3·19) we note r_l satisfies the same relations as (A·15) and (A·16). Then, $\bar{r}_{2^m} = r_{2^m}$ for $(m \geq 1)$. Therefore, $r_l = \bar{r}_l$ follows for $l \geq 1$. Finally, from the relation (A·13) we have $\bar{r}_0 = \alpha = r_0$. Therefore, we obtain $\hat{G}_{m-1}(a) = G_{m-1}(a)$. That is, the relation (3·32) holds.

Appendix B
—— *Proof of* $\lim_{m\to\infty} h_m = 0$ ——

In this appendix, we prove

$$\lim_{m\to\infty} h_m = 0,$$

where h_m is

$$h_m = \bar{h}_m(1 + a_c^{-2^m}\gamma^{\zeta_m - 2\hat{\zeta}_m}\frac{q_m}{b}) + a_c^{-2^m}\gamma^{\zeta_m-2\hat{\zeta}_m}\frac{q_m}{b},$$

$$q_m = -\frac{1}{1 - a_c^{-2^m}\gamma^{-\hat{\zeta}_m}}(\alpha - \gamma s_{2^m} + \gamma^{2\hat{\zeta}_m}\sum_{l=0}^{\infty} r_{2^{m+1}+l}a_c^{-l}).$$

To show this, we only have to prove that

$$\lim_{m\to\infty} a_c^{-2^m}\gamma^{\zeta_m-2\hat{\zeta}_m}q_m = 0.$$

First we estimate the series $S \equiv \sum_{l=0}^{\infty} r_{2^{m+1}+l} a_c^{-l}$.
Let τ_l be the number of 1 in $s_1, s_2, \cdots, s_l$. Then, r_l is

$$r_l = \gamma s_l [\prod_{j=1}^{l} \{1-(1+\gamma)s_j\}]^{-1} = \gamma s_l (-\gamma)^{-\tau_l}. \qquad \text{(B·1)}$$

Thus,

$$|r_l| \le \gamma^{1-\tau_l}.$$

Let $\tilde{\tau}_l$ be the number of 1 in $s_{2^{m+1}+1}, s_{2^{m+1}+2}, \cdots, s_{2^{m+1}+l}$. Then, for $l \ge 0$ we obtain

$$\tau_{2^{m+1}+l} = \hat{\zeta}_{m+1} + \tilde{\tau}_l, \;\; \tilde{\tau}_l \le l, \;\; \tilde{\tau}_0 \equiv 0.$$

Then,

$$|S| \le \sum_{l=0}^{\infty} \gamma^{1-\hat{\zeta}_{m+1}-\tilde{\tau}_l} a_c^{-l}.$$

Since $a_c \gamma > 1$, we obtain

$$|S| \le \frac{a_c}{a_c - 1} \gamma^{1-\hat{\zeta}_{m+1}}, \quad \text{for } \gamma \ge 1,$$
$$|S| \le \frac{a_c \gamma}{a_c \gamma - 1} \gamma^{1-\hat{\zeta}_{m+1}}, \quad \text{for } \gamma < 1.$$

Thus for any γ,

$$|S| \le \text{const.} \gamma^{-\hat{\zeta}_{m+1}}.$$

From the following relation

$$\zeta_m - \hat{\zeta}_{m+1} = -\hat{\zeta}_m - \frac{1+(-1)^{m-1}}{2},$$

we get

$$a_c^{-2^m} \gamma^{\zeta_m} |S| \le \text{const.} a_c^{-2^m} \gamma^{-\hat{\zeta}_m} \gamma^{-\frac{1+(-1)^{m-1}}{2}}.$$

Further,

$$\zeta_m - 2\hat{\zeta}_m = -\hat{\zeta}_m - \frac{1+(-1)^m}{2},$$

and then

$$a_c^{-2^m} \gamma^{\zeta_m - 2\hat{\zeta}_m} = a_c^{-2^m} \gamma^{-\hat{\zeta}_m} \gamma^{-\frac{1+(-1)^m}{2}}.$$

Thus, since $a_c^{-2^m} \gamma^{-\hat{\zeta}_m}$ tends to 0 as $m \to \infty$, we obtain

$$\lim_{m \to \infty} a_c^{-2^m} \gamma^{\zeta_m - 2\hat{\zeta}_m} q_m = 0.$$

Appendix C

—— *Proofs of* $a_{p,max} = \frac{p-1}{p} a_M$ *and* $x_p(a_{p,max}) > \beta$ *for* $p \geq 3$ ——

In this appendix, we prove that for the asymmetric case, $x_p = \gamma a^{p-1}(1 - a\alpha)$ has the maximum value at $a = a_{p,max} \equiv \frac{p-1}{p} a_M$ and $x_p(a_{p,max}) > \beta$ for $p \geq 3$. Similar results for the symmetric case is obtained by putting $\gamma = 1$.

Differentiating $x_p(a)$ w.r.t. a, we get

$$x_p'(a) = \gamma a^{p-2}(p-1)(1 - \frac{p}{p-1} a\alpha). \tag{C·1}$$

Then, at $a = a_{p,max} \equiv \frac{p-1}{p} a_M$, $x_p(a)$ has the maximum value

$$x_p(a_{p,max}) = \frac{\gamma}{p}(\frac{p-1}{p})^{p-1} a_M^{p-1} = \frac{1}{p(1-\beta)}(\frac{p-1}{p})^{p-1}\alpha^{2-p}. \tag{C·2}$$

Since $0 < \alpha < \beta < 1$ and $p \geq 3$,

$$x_p(a_{p,max}) > \frac{1}{p(1-\beta)}(\frac{p-1}{p})^{p-1}\beta^{2-p} = \frac{1}{p}(\frac{p-1}{p})^{p-1}\frac{\beta}{g(\beta)}, \tag{C·3}$$

where $g(\beta) \equiv \beta^{p-1}(1-\beta) > 0$. $g(\beta)$ has the maximum value at $\beta = \frac{p-1}{p}$ and

$$g(\frac{p-1}{p}) = \frac{1}{p}(\frac{p-1}{p})^{p-1}. \tag{C·4}$$

Thus,

$$x_p(a_{p,max}) > \frac{1}{p}(\frac{p-1}{p})^{p-1}\frac{\beta}{g(\frac{p-1}{p})} = \beta. \tag{C·5}$$

Appendix D

—— *Proof of* $H(x_1)H(x_2)\cdots H(x_{p-1}) = RL\cdots L$ *for* $a_{p,0} \leq a$ ——

In this appendix, we prove that for $a_{p,0} \leq a$, the symbolic sequence for $x_1, x_2, \cdots, x_{p-1}$ is $RL\cdots L$. We prove this in the asymmetric case. We obtain the results for the symmetric case by putting $\gamma = 1$.

First, we prove the following relation,

$$a_{p,0} > \frac{\beta}{\alpha}. \tag{D·1}$$

For $p \geq 3$, the following relation holds,

$$x_p(\frac{\beta}{\alpha}) = (\frac{\beta}{\alpha})^{p-2}\beta > \beta. \tag{D·2}$$

Since $x_p(a) = \gamma a^{p-1}(1 - a\alpha)$ is a strictly decreasing continuous function for $a_{p,max} \leq a$, and $a_{p,max} < a_{p,0}$, we obtain $a_{p,0} > \frac{\beta}{\alpha}$.
Now, let us fix a such as $a \geq a_{p,0}$. Then, from the relation (D·1)

$$x_1 = a\alpha \geq a_{p,0}\alpha > \beta.$$

Thus, $H(x_1) = R$. Then,

$$x_2 = \gamma a(1 - x_1) = \frac{\gamma a^{p-1}(1 - a\alpha)}{a^{p-2}} = \frac{x_p}{a^{p-2}} \leq \frac{\beta}{a^{p-2}} \leq \frac{\beta}{a_{p,0}^{p-2}} < \frac{\beta}{a_{p,0}} < \alpha.$$

Thus, $x_2 \in I_L$. Now, let us assume $x_n < \beta$ for $2 \leq n \leq p-2$. Then,

$$x_{n+1} = ax_n = a^n\gamma(1 - a\alpha) = \frac{a^{p-1}\gamma(1 - a\alpha)}{a^{p-1-n}} < \frac{\beta}{a^{p-1-n}} < \frac{\beta}{a} \leq \frac{\beta}{a_{p,0}} < \alpha.$$

Therefore, $x_{n+1} \in I_L$. Thus,

$$x_2 < x_3 < \cdots < x_{p-1} < \alpha.$$

Q.E.D.

Appendix E

—— *Proof of* $a_c \leq a_{3,0} < a_{4,0} < \cdots < a_{p,0} < a_{p+1,0} < \cdots < a_M$ ——

In this appendix, we prove the following relations,

$$a_c \leq a_{3,0} < a_{4,0} < \cdots < a_{p,0} < a_{p+1,0} < \cdots < a_M.$$

We only have to prove it for the asymmetric case because $\gamma = 1$ reduces to the symmetric case.

Let us estimate $x_{p+1}(a_{p,0})$ for $p \geq 3$,

$$x_{p+1}(a_{p,0}) = a_{p,0}x_p(a_{p,0}) = a_{p,0}\beta > \beta.$$

Since $x_{p+1}(a)$ is decreasing for $a_{p+1,max} \leq a$ and $a_{p+1,max} < a_{p+1,0}$, $a_{p,0} < a_{p+1,0}$ follows.
Let us prove the first relation $a_c \leq a_{3,0}$. For $a_{3,0} < a < a_{3,1}$, there is a stable periodic three solution. If $a_c > a$, there is a stable periodic solution with period 2^m with some integer m, and no other stable solution. Thus, $a_c \leq a_{3,0}$. Q.E.D.

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